Characters of Representations of Reductive p-adic Groups

Characters of Representations of Reductive p-adic Groups

Proefschrift ter verkrijging van de graad van doctor
aan de Radboud Universiteit Nijmegen
op gezag van de rector magnificus prof. dr. J.H.J.M. van Krieken,
volgens besluit van het college van decanen
in het openbaar te verdedigen op vrijdag 17 maart 2017
om 12:30 uur precies

door

Julius Witte

geboren op 10 januari 1989
te Utrecht

Promotor:
Prof. dr. G.J. Heckman

Copromotor:
Dr. M.S. Solleveld

Manuscriptcommissie:

Prof. dr. B.J.J. Moonen	
Prof. dr. J. Adler	American University
Prof. dr. A.M. Aubert	Université Pierre-et-Marie-Curie
Prof. dr. R. Meyer	Georg-August-Universität Göttingen
Prof. dr. E. Opdam	Universiteit van Amsterdam

uitgever, *plaats*

Contents

Chapter 1

Introduction

In this thesis we study the geometry of reductive p-adic groups, with applications in representation theory in mind. We will consider the characters of representations, the nilpotent orbits in the Lie algebra and supercuspidal representations of GL_N. Before we give an outline of these topics, first some general comments on reductive p-adic groups.

The, up to now established, theory of reductive p-adic groups has two major dividing points: p-adic fields vs local fields of positive characteristic and local vs global.
The theory on reductive groups defined over p-adic fields is in a more established state than the one over local fields of positive characteristic. Since the characteristic of p-adic fields is 0, well-known methods in real or complex reductive groups can be generalized more easily than for non-Archimedean fields with positive characteristic. The theory established for p-adic fields should more or less also hold for local fields of positive characteristic.
The local vs global division is more intrinsic. In the local approach one fixes a reductive group and a non-Archimedean local field, whereas in the global approach the non-Archimedean local field varies. It turns out that looking at multiple fields together (see adèles), one can prove statements for most and sometimes all of these fields. This is one of the reasons why some theorems have to assume that the characteristic of the (residue) field is large enough. Such theorems are also used to go around the other way: from local to global.
This thesis focuses on the local fields of positive characteristic and the local side of the theory. The main goal is to prove theorems, known for p-adic fields, without any assumptions on the characteristic of the non-Archimedean local field. In this thesis some small steps are made in this direction.

1.1 Outline

This thesis consists of six chapters. The first two are this introduction and a background. The other four chapters can be read separately. These four chapters form the heart of the thesis and consist of slightly related topics. Now we briefly describe these topicsr. A more detailed introduction can be found in the beginning of each of these chapters.

In Chapter 3 we calculate an upper bound for characters of finite length representations on a regular semisimple element γ contained in the centralizer of a maximal split torus. This is done by looking at the building of the reductive p-adic group. The key result is an estimate of the number of fixed points of γ in a particular part of the building. This estimate is used to get an upper bound for the character and the Weyl integration formula. Together with the Weyl integration formula and some calculations on tori these estimates are good enough to show that the trace is locally summable on the conjugation orbit of the torus containing both γ and a maximal split torus.

Chapter 4 is about the geometry of the nilpotent orbits. The main focus is on split reductive groups. We will determine when there are only finitely many nilpotent orbits, when the orbits are separable and when Howe's conjecture on the Lie algebra holds in terms of the root datum and the characteristic of the field. In both the proof of Howe's conjecture in certain cases and the counterexamples to Howe's conjecture the nilpotent orbits play a key role.

In Chapter 5 we discuss characters of supercuspidal representations of GL_N and the degree of these representations. This is a first attempt to prove for GL_N a conjecture of Kim, Shin and Templier: for a fixed semisimple regular element γ, the absolute value of the character of a supercuspidal representation in γ divided by the degree of the representation goes to 0 when the degree goes to infinity. We prove this conjecture for certain collections of supercuspidal representations. The conjecture also holds when N is prime or $N > 8$ and N is the product of two primes.

Chapter 6 is a short chapter also concerning fixed points in the Bruhat-Tits building. We will show that for every reductive group G, there exists a $C > 0$ such that for every compact element $g \in G$ and x in the building, there exists a p in the building fixed by g such that $d(x, p) \leq Cd(x, gx)$. This theorem was conjectured by Tsai.

Chapter 2

Background

2.1 Non-Archimedean local fields

The theory of non-Archimedean local fields can be found in for example [Ser68] and [Gou97]. In this section we give a brief introduction to non-Archimedean local fields and fix some notation.

Let $\mathbb{F}$ be a field. A *norm* $|\cdot|$ on $\mathbb{F}$ is a function $|\cdot| : \mathbb{F} \to \mathbb{R}_{\geq 0}$ such that for all $a, b \in \mathbb{F}$:

$$\begin{aligned} |0| &= 0, \\ |ab| &= |a||b|, \\ |a+b| &\leq |a| + |b|. \end{aligned}$$

If moreover $|a+b| \leq \max(|a|, |b|)$, then the norm is called a *non-Archimedean norm.*
A *non-Archimedean local field* is a field $\mathbb{F}$ together with a norm $|\cdot|$ on $\mathbb{F}$ such that $(\mathbb{F}, |\cdot|)$ is locally compact and not discrete as a topological space.

Example 2.1. *Let $\mathbb{Q}$ be the rational numbers. Let p be a prime number. Define the valuation: $\nu_p : \mathbb{Q} - \{0\} \to \mathbb{Z}$, by the following equality:*

$$q = p^{\nu_p(q)} \frac{a}{b}, \text{ with } a, b \in \mathbb{Z},\ p \not| a, b.$$

The valuation ν_p is called the p-adic valuation on $\mathbb{Q}$. The p-adic norm $|\cdot|_p$ is defined as follows:

$$|q|_p := \begin{cases} 0 & \text{if } q = 0, \\ p^{-\nu_p(q)} & \text{if } q \neq 0. \end{cases}$$

The completion of $\mathbb{Q}$ with respect to this norm is called the p-adic numbers *and denoted by $\mathbb{Q}_p$. Now we can extend the valuation ν_p and norm $|\cdot|_p$ to $\mathbb{Q}_p$. With abuse of notation we denote these extensions by ν_p and $|\cdot|_p$. The field $\mathbb{Q}_p$ with the norm $|\cdot|_p$ is a non-Archimedean local field.*

Example 2.2. *Let p be a prime number and q a power of p. Let $\mathbb{F}_q$ be the field with q elements. Let*

$$\mathbb{F}_q((X)) := \left\{ \sum_{i=-k}^{\infty} c_i X^i : c_i \in \mathbb{F}_q, k \in \mathbb{N} \right\}.$$

Define on this field the following valuation ν_X:

$$\nu_X\left(\sum_{i=-k}^{\infty} c_i X^i\right) = \max\{i \in \mathbb{Z} \mid c_j = 0 \textit{ for all } j < i\}.$$

The norm $|\cdot|$ is defined as follows:

$$|a| := \begin{cases} 0 & \textit{if } a = 0, \\ q^{-\nu_X(a)} & \textit{if } a \neq 0. \end{cases}$$

The field $\mathbb{F}_q((X))$ together with the norm $|\cdot|$ is a non-Archimedean local field.

Theorem 2.3. *If $\mathbb{F}$ is a non-Archimedean local field, then exactly one of the following two statements holds:*

1. $\mathbb{F}$ *is a finite field extension of $\mathbb{Q}_p$ for some prime p.*
2. $\mathbb{F}$ *is isomorphic to $\mathbb{F}_q((X))$ for some prime power q.*

Unless otherwise stated, $\mathbb{F}$ will be a non-Archimedean local field throughout the remainder of this thesis.

From now on, when we say $\mathbb{F}$ is a non-Archimedean local field, we denote its valuation by ν and its norm by $|\cdot|$. We define the following objects attached to a p-adic field $\mathbb{F}$:

$$\begin{aligned} \mathcal{O} &:= \{x \in \mathbb{F} \mid \nu(x) \geq 0\}, \\ \mathcal{O}^\times &:= \{x \in \mathbb{F} \mid \nu(x) = 0\}, \\ \mathfrak{p} &:= \{x \in \mathbb{F} \mid \nu(x) > 0\}, \\ \kappa_{\mathbb{F}} &:= \mathcal{O}/\mathfrak{p}, \\ q &:= |\kappa_{\mathbb{F}}|, \\ p &:= \operatorname{char} \mathbb{F}, \end{aligned}$$

where $\mathcal{O}$ is called the *ring of integers* of $\mathbb{F}$, $\mathcal{O}^\times$ the *group of units* of $\mathcal{O}$, $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$, $\kappa_{\mathbb{F}}$ is the *residue field* of $\mathbb{F}$, q is the order of the residue field of $\mathbb{F}$ and p the characteristic of the field $\mathbb{F}$. A *uniformizer* of $\mathbb{F}$ is an element $\varpi \in \mathfrak{p}$ with $\varpi\mathcal{O} = \mathfrak{p}$. From now on ϖ will always denote a uniformizer of $\mathbb{F}$. Unless otherwise stated, we assume that $\operatorname{Im}(\nu) = \mathbb{Z}$ and $|x| = q^{-\nu(x)}$ for all $x \in \mathbb{F}^\times$. When there are multiple p-adic fields, we distinguish the objects belonging to $\mathbb{F}$ by putting the subscript $\mathbb{F}$ on the right of these objects: $\mathcal{O}_{\mathbb{F}}, \mathfrak{p}_{\mathbb{F}}, \varpi_{\mathbb{F}}$, etc..

Let $\mathbb{E}$ be a finite field extension of $\mathbb{F}$. Then $\mathbb{E}$ is also a non-Archimedean local field. Moreover the residue field of $\mathbb{F}$ can be viewed as a subfield of the residue field of $\mathbb{E}$. To be more precise, $\kappa_{\mathbb{F}}$ is embedded in $\kappa_{\mathbb{E}}$ by the following map:

$$a + \mathfrak{p}_{\mathbb{F}} \mapsto a + \mathfrak{p}_{\mathbb{E}},$$

with $a \in \mathcal{O}_{\mathbb{F}}$. The field index $f = f(\mathbb{E} : \mathbb{F}) = [\kappa_{\mathbb{E}} : \kappa_{\mathbb{F}}]$ is called the *residue degree* of $\mathbb{E}$ and $\mathbb{F}$. Let e be such that $\varpi_{\mathbb{E}}^e \mathcal{O}_{\mathbb{E}} = \varpi_{\mathbb{F}} \mathcal{O}_{\mathbb{E}}$. The number $e = e(\mathbb{E} : \mathbb{F})$ is called the *ramification index* of the field extension. Now

$$[\mathbb{E} : \mathbb{F}] = ef.$$

A field extension $\mathbb{E} : \mathbb{F}$ is called *unramified* if $e = 1$, *tamely ramified* if $\gcd(e, q) = 1$, *wildly ramified* if $\gcd(e, q) \neq 1$ and *totally ramified* if $f = 1$.
If $\mathbb{E} : \mathbb{F}$ is a finite field extension, then there exists an intermediate field $\mathbb{E}_{nr}$ which is the maximal unramified field extension of $\mathbb{F}$. Then $\mathbb{E} : \mathbb{E}_{nr}$ is totally ramified and $\mathbb{E}_{nr} : \mathbb{F}$ is unramified.

2.2 Reductive groups

In this section we introduce the notion of a reductive group and the most relevant objects attached to it. First we discuss reductive groups over algebraically closed fields, later we will discuss reductive groups defined over non-Archimedean local fields. See for example [Spr98] for proofs and a detailed account on reductive groups.

A *linear algebraic group* $\mathcal{G}$ is an affine algebraic variety which is also a group such that multiplication and inversion are algebraic morphisms.

Example 2.4. *Let k be an algebraically closed field. Then $(k, +)$ is an algebraic group.*

Example 2.5. *Let k be an algebraically closed field.*
We define the linear algebraic group $GL_n(k)$ as follows:
$GL_n(k)$ is a subvariety of $k \times k^{n\times n}$. Let T_{ij} be the coordinate function defined by $T_{ij}(x, M) = M_{ij}$, for all $x \in k$ and $M \in k^{n\times n}$. Let $D : k \times k^{n\times n} \to k$ be the function defined by $D(x, M) = x$.
Let $k[D^{-1}, T_{ij}]$ be the k-algebra belonging to the affine variety $k \times k^{n\times n}$. Let $GL_n(k)$ be the subvariety defined by

$$\det \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \cdot D^{-1} = 1.$$

We view the points of $GL_n(k)$ as invertible $(n \times n)$-matrices by ignoring the first coordinate. The group structure of $GL_n(k)$ is inherited from the invertible $(n \times n)$-matrices. Since $k^\times \cong GL_1(k)$, also $k^\times$ is an algebraic group.

Example 2.6. *Let D_n be the subgroup of GL_n consisting of diagonal matrices. It is a Zariski-closed subgroup of GL_n, hence a linear algebraic group.*

Let $X^*(\mathcal{G})$ be the set of *characters* of G, i.e., algebraic group homomorphisms $\chi : \mathcal{G} \to k^\times$. Let $X_*(\mathcal{G})$ be the set of *cocharacters* of G, i.e., algebraic group homomorphisms $X : k^\times \to G$. There exists a natural pairing $\langle\ ,\ \rangle$ between the characters and cocharacters. Let $\chi \in X^*(\mathcal{G})$ and $X \in X_*(\mathcal{G})$. The only algebraic group automorphisms of $k^\times$ are $\phi_z : x \mapsto x^z$, for $z \in \mathbb{Z}$. Let $z \in \mathbb{Z}$ be such that $\chi \circ X = \phi_z$. Then we define $\langle X, \chi\rangle := z$.

A *Borel subgroup* of $\mathcal{G}$ is a closed, connected, solvable, subgroup of $\mathcal{G}$, which is maximal for these properties. Every linear algebraic group has a Borel subgroup.
A *torus* $\mathcal{T}$ is a linear algebraic group isomorphic to D_n for some $n \in \mathbb{N}$. In this case the

pairing $\langle\ ,\ \rangle$ defines an isomorphism between $X^*(\mathcal{T})$ and $\mathrm{Hom}_{\mathbb{Z}}(X_*(\mathcal{T}), \mathbb{Z})$.
The maximal closed connected normal solvable subgroup of $\mathcal{G}$ is called the *radical* of $\mathcal{G}$ and will be denoted by $R(\mathcal{G})$.
The *unipotent radical* $R_u(\mathcal{G})$ is the linear algebraic group consisting of the unipotent elements of $R(\mathcal{G})$.
The group $\mathcal{G}$ is called *semisimple* if $R(\mathcal{G}) = \{e\}$ and *reductive* if $R_u(\mathcal{G}) = \{e\}$.
A *maximal torus* of $\mathcal{G}$ is a torus of $\mathcal{G}$ which is not contained in a bigger subtorus of $\mathcal{G}$.

Let $\lambda(x) : k[\mathcal{G}] \to k[\mathcal{G}]$ be the linear map corresponding with the left-multiplication by x on $\mathcal{G}$. The *Lie algebra* $\mathfrak{g}$ of $\mathcal{G}$ is the set of k-derivations of $k[\mathcal{G}]$ in $k[\mathcal{G}]$, which are invariant under the action $\lambda(x)$. The Lie algebra is also isomorphic to $T_e\mathcal{G}$, the tangent space of $\mathcal{G}$ at e. For $x \in \mathcal{G}$ define $\mathrm{Ad}(x) : \mathcal{G} \to \mathcal{G}$ by $\mathrm{Ad}(x)(g) := xgx^{-1}$. This is an algebraic homomorphism with $\mathrm{Ad}(x)(e) = e$. Thus it corresponds with a linear action on $T_e\mathcal{G} = \mathfrak{g}$, which we also denote by $\mathrm{Ad}(x)$. This is called the *adjoint action* of $\mathcal{G}$ on $\mathfrak{g}$.

Let $\mathcal{G}$ be a reductive group. Let $\mathcal{T}$ be a maximal torus of $\mathcal{G}$. Then $\mathcal{T}$ acts on $\mathfrak{g}$ by the adjoint action. Since $\mathcal{T}$ is a set of commuting semisimple elements, $\mathfrak{g}$ decomposes into a direct sum of weight spaces:

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(\mathcal{T})} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \forall (t \in \mathcal{T})\ \mathrm{Ad}(t)x = \alpha(t)x\}.$$

A *root* of $\mathcal{G}$ and $\mathcal{T}$ is a nontrivial character α of $\mathcal{T}$ such that $\mathfrak{g}_\alpha \neq 0$. The set of roots of $\mathcal{G}$ and $\mathcal{T}$ is denoted by $R(\mathcal{G}, \mathcal{T})$. Then

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $\mathfrak{t}$ is the Lie algebra of $\mathcal{T}$ and $R := R(\mathcal{G}, \mathcal{T})$. Moreover, the vector spaces $\mathfrak{g}_\alpha$ are one-dimensional. For $\alpha \in R$ there exists a homomorphism of algebraic groups $u_\alpha : k \to \mathcal{G}$ such that $\mathrm{Im}\ \mathrm{d}u_\alpha = \mathfrak{g}_\alpha$ and $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for all $x \in k$ and $t \in \mathcal{T}$.

The *Weyl group* of $(\mathcal{G}, \mathcal{T})$ is denoted as W and is defined by $W = N_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$. This group W acts on $X^*(\mathcal{T})$ via the conjugation action of W on $\mathcal{T}$.
Define $\mathcal{G}_\alpha$ to be the centralizer of the subtorus $(\ker \alpha)^0$ of $\mathcal{T}$. Then $N_{\mathcal{G}_\alpha}(\mathcal{T})/\mathcal{T}$ has order two. Let s_α be its generator. Then $s_\alpha \in W$ and hence acts on $X^*(\mathcal{T})$. There exists a unique $\alpha^\vee \in X_*(\mathcal{T})$ such that for all $x \in X^*(\mathcal{T})$:

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle\, \alpha.$$

Define $R^\vee := \{\alpha^\vee : \alpha \in R\}$.

The *root datum* of $(\mathcal{G}, \mathcal{T})$ is the quadruple $(X^*(\mathcal{T}), R, X_*(\mathcal{T}), R^\vee)$.

Theorem 2.7. *[Spr98, Theorem 9.6.2 & Theorem 10.1.1] Let $\mathcal{G}_i$ be a reductive group and $\mathcal{T}_i$ a maximal torus of $\mathcal{G}_i$ for $i = 0, 1$. Let $\Psi_i = (X^*(\mathcal{T}_i), R_i, X_*(\mathcal{T}_i), R_i^\vee)$ be the root datum*

of $(\mathcal{G}_i, \mathcal{T}_i)$. If Ψ_0 is isomorphic to Ψ_1, then $\mathcal{G}_0$ is isomorphic to $\mathcal{G}_1$.
If $(X, R, X^\vee, R^\vee)$ is a root datum, then there exists a reductive group $\mathcal{G}$ and a maximal torus $\mathcal{T}$ of $\mathcal{G}$ such that $(X, R, X^\vee, R^\vee)$ is isomorphic to $(X^*(\mathcal{T}), R, X_*(\mathcal{T}), R^\vee)$.

For now, here ends the discussion of the theory on reductive groups over algebraically closed fields. We will follow with reductive groups over non-Archimedean local fields.

Let $\mathbb{F}$ be a non-Archimedean local field and k an algebraic closure of $\mathbb{F}$. Let $\mathcal{G}$ be a k-reductive group defined over $\mathbb{F}$. Let $G := \mathcal{G}(\mathbb{F})$ be the group of $\mathbb{F}$-rational points of $\mathcal{G}$. This thesis is about the geometry of G.

From now on, if we write that G is a linear algebraic group over $\mathbb{F}$, then we imply that we have chosen a linear algebraic group $\mathcal{G}$ defined over $\mathbb{F}$ such that $\mathcal{G}(\mathbb{F}) = G$.

A *reductive p-adic group* is a reductive group over a non-Archimedean local field.

Example 2.8. *$GL_n(\mathbb{F})$ is a reductive group defined over $\mathbb{F}$.*
$D_n(\mathbb{F}) \equiv (\mathbb{F}^\times)^n$ is a torus defined over $\mathbb{F}$. Tori over $\mathbb{F}$ isomorphic to $(\mathbb{F}^\times)^n$ are called split tori.

Every linear algebraic group over $\mathbb{F}$ can be algebraically embedded in $GL_n(\mathbb{F})$ for some $n \in \mathbb{N}$. Thus a linear algebraic group over $\mathbb{F}$ is more or less a subgroup of $GL_n(\mathbb{F})$ which is also the zero locus of finitely many polynomials in $\mathbb{F}[T_{ij}]$. The tricky part is that these polynomials must generate a radical ideal in $k[T_{ij}]$. See [Spr98, §12.1.6] for some instructive counterexamples.

2.3 Representations and Hecke algebras

In this section, we define the representations which we will study and show that these are representations of an algebra called the Hecke algebra of G. See [Ren10] and [Car79] for proofs and more details.

Let G be a linear algebraic group over $\mathbb{F}$. On $GL_n(\mathbb{F})$ we take the topology from $\mathbb{F}^{n\times n}$. The topology of the group G is the induced topology from an embedding of G in $GL_n(\mathbb{F})$.

Let V be a complex vector space and (π, V) a representation of G. For all subsets $X \subset G$, define
$$V^X := \{v \in V \mid xv = v \text{ for all } x \in X\}.$$
We call (π, V) *smooth* if for every vector $v \in V$ there exists a compact open subgroup $K \subset G$ such that $v \in V^K$. If moreover V^K is finite dimensional for all compact open subgroups K of G, then (π, V) is called *admissible*.

Example 2.9. *Let $G = \mathbb{F}^\times$. Let V be the $\mathbb{C}$-vector space with basis $\{e_z : z \in \mathbb{Z}\}$. Define the representation π by*
$$\pi(x)e_z = e_{z+\nu(x)}, \text{ for all } x \in \mathbb{F}^\times, z \in \mathbb{Z}.$$

Then (π, V) is a smooth representation. However it is not admissible, because $V^{\mathcal{O}^\times} = V$. Let $G = GL_n(\mathbb{F})$ and $V = \mathbb{C}$. Define $\pi(g) = |\det g|$. Then (π, V) is an admissible representation.

We will now define the Hecke algebra of G. The Hecke algebra plays the same role as the group ring in the representation theory of finite groups.

Let μ be a Haar measure on G. Since G is reductive μ is both left and right invariant. Let K be a compact open subgroup of G. Define $\mathscr{H}(G, K)$ to be the set of compactly supported functions $f : G \to \mathbb{C}$ satisfying:

$$f(kgk') = f(g), \text{ for } g \in G, \ k, k' \in K.$$

Define the multiplication on $\mathscr{H}(G, K)$ as follows:

$$(f_1 * f_2)(g) = \int_{x \in G} f_1(x) f_2(x^{-1} g) \mathrm{d}x.$$

This turns $\mathscr{H}(G, K)$ into an associative algebra over $\mathbb{C}$. It has the following function as unit:

$$e_K(x) := \begin{cases} 0 & \text{if } x \notin K, \\ \mu(K)^{-1} & \text{if } x \in K. \end{cases}$$

Define the *Hecke algebra* of G by

$$\mathscr{H}(G) = \bigcup_{\substack{K < G \\ \text{open compact subgroup}}} \mathscr{H}(G, K).$$

Since $\mathscr{H}(G, K) \subset \mathscr{H}(G, K')$ whenever $K' < K$, the multiplication $*$ is defined on $\mathscr{H}(G)$. Therefore, $\mathscr{H}(G)$ is an associative algebra over $\mathbb{C}$.

Let (π, V) be a smooth representation of G. We define a representation of $\mathscr{H}(G)$ on V also denoted by π as follows:

$$\pi(f)v = \int_G f(g)\pi(g)v \mathrm{d}g.$$

This turns V into a nondegenerate $\mathscr{H}(G)$-module, i.e., $\mathscr{H}(G)V = V$.

Conversely, if V is a nondegenerate $\mathscr{H}(G)$-module, we define a representation π of G on V as follows: Let $v \in V$. Since V is nondegenerate, there exists an open compact K such that $e_K \cdot v = v$. Define $\pi(g)v := e_{gK} \cdot v$, where

$$e_{gK}(x) := \begin{cases} 0 & \text{if } x \notin gK, \\ \mu(K)^{-1} & \text{if } x \in gK. \end{cases}$$

Then (π, V) is a smooth representation of G.

Moreover, the procedures of going from smooth representations to nondegenerate modules and vice versa can be turned into functors between the two categories. These functors are each other's inverse.

Define the following set of functions on a totally disconnected topological space X:

$$C_c^\infty(X) := \{f : G \to \mathbb{C} \mid f \text{ is locally constant and has compact support}\}.$$

As sets of functions on G, $\mathscr{H}(G) = C_c^\infty(G)$.

2.4 HC-Theorem

We will now define the main object of this thesis: the character of an admissible representation.

Let G be a reductive p-adic group and let (π, V) be an admissible representation.

We want to define the value of the character of π at g to be the trace of $\pi(g)$ on V. However, since V may be infinite dimensional, the trace of $\pi(g)$ on V may be ill-defined. To get a better definition we first pass through the Hecke algebra of G.

Let $f \in \mathscr{H}(G, K)$, then $\pi(f)V \subset V^K$ ($e_K * f = f$ and $e_K V = V^K$). The trace of $\pi(f)$ is well-defined because V^K is finite dimensional.
Let G' be the set of regular semisimple elements of G. The set G' is open and dense in G.

Theorem 2.10 (Harish-Chandra)**.** *There exists a locally constant function $\theta_\pi : G' \to \mathbb{C}$ such that for all $f \in C_c^\infty(G')$:*

$$\mathrm{tr}(\pi(f), V) = \int_G \theta_\pi(g) f(g) \mathrm{d}g.$$

The function θ_π is called the *character* of π. We extend θ_π to all of G by setting $\theta_\pi(g) := 0$ for all $g \in G \backslash G'$.
Let $g \in G$ be semisimple. Define $D(g)$ to be the *Harish-Chandra D-function*: Let T be a maximal torus containing g. Define $D(g) := \prod_{\alpha \in R(G,T)} (\alpha(g) - 1)$, where $R(G, T)$ is the root system of T and G. Let $\lambda(g)$ be such that $q^{\lambda(g)} = |D(g)|$.

Theorem 2.11 (Harish-Chandra)**.** *Assume char $\mathbb{F} = 0$.*
The function θ_π is locally integrable on G and, for every $f \in C_c^\infty(G)$,

$$\mathrm{tr}(\pi(f), V) = \int_G f(g) \theta_\pi(g) \mathrm{d}g.$$

The function $|D(g)|^{\frac{1}{2}} \theta_\pi(g)$ is locally bounded.

One of the goals of this thesis is to generalize the HC-Theorem to non-Archimedean local fields of positive characteristic. There has been progress on generalizing this theorem in two directions. It has been shown that the theorem also holds for particular groups, e.g., SL_n [Lem96] and GL_n [Lem05, Rod85]. Also, for every group G defined over $\mathbb{Z}$ there is an N such that if $p > N$, then the theorem holds [CGH14]. In both [Rod85] and [CGH14] one more or less generalizes the proof given by Harish-Chandra in [HC99] to fields of positive characteristic. For each step in the proof one tries to generalize this step to positive characteristic and/or keep track of the assumptions made, see for example [DeB02a].

In this thesis we consider two methods, [HC70] & [HC99], used by Harish-Chandra in order to prove this theorem. Both methods use that char $\mathbb{F} = 0$. First we consider the argument of [HC70].

For $\omega \subset G$, define ${}^G\omega := \{gwg^{-1} : w \in \omega, g \in G\}$.

If π is a cuspidal representation, Harish-Chandra [HC70] proves (in characteristic 0) that θ_π is locally summable. His proof consists of four steps for all maximal tori T:

1. For every $g \in G$ there exist a compact neighborhood ω of g, a $C \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$ such that for all $\gamma \in \omega$
$$|\theta_\pi(\gamma)| \leq C|\lambda(\gamma)|^n|D(\gamma)|^{-\frac{1}{2}}.$$
2. For all $\epsilon \geq 0$
$$\int_{{}^GT} |D(x)|^{-\frac{1}{2}-\epsilon} f(x)\mathrm{d}x = |N_G(T)/T|^{-1} \int_T |D(t)| \int_{G/T} |D(t)|^{-\frac{1}{2}-\epsilon} f(gtg^{-1})\mathrm{d}g\mathrm{d}t.$$
3. For every $f \in C_c^\infty(G)$ there exists a $C \in \mathbb{R}_{>0}$ such that for all regular $t \in T$
$$\int_{G/T} f(gtg^{-1})\mathrm{d}g \leq C|D(t)|^{-\frac{1}{2}}.$$
4. For small $\epsilon \geq 0$ the function $|\lambda(t)|^n|D(t)|^{-\epsilon}$ is locally summable on T.

The local summability of the character on G follows from these four statements because, when the characteristic is 0, there are only finitely many conjugacy classes of maximal $\mathbb{F}$-tori:

$$\begin{aligned}\int_{{}^GT} |f\theta_\pi|\mathrm{d}g &\leq \int_{{}^GT} C_{\theta_\pi}|\lambda(g)|^n|D(g)|^{-\frac{1}{2}}|f(x)|\mathrm{d}x \\ &= |W|^{-1}\int_T C_{\theta_\pi}|D(t)|^{\frac{1}{2}}\int_{{}^GT}|\lambda(gtg^{-1})|^n|f(gtg^{-1})|\mathrm{d}g\mathrm{d}t \\ &\leq \frac{C_{\theta_\pi}C_f}{|W|}\int_T |\lambda(t)|^n\mathrm{d}t \leq C_\epsilon \int_T |D(t)|^{-\epsilon}\mathrm{d}t.\end{aligned} \tag{2.1}$$

In Chapter 3 we partially generalize step 1 and 3, and prove step 4. Step 2 will be proved in Theorem 2.16.

In [HC99] the second method is described. We will follow its preface and introduction. We first need to introduce some concepts and some notations before we can start with the summary.

Let B be a nondegenerate, G-invariant bilinear form on $\mathfrak{g}$. Such a form B exists because G is a reductive group over a field of characteristic 0. Let $\psi : \mathbb{F} \to \mathbb{C}^\times$ be an additive character. For $f \in C_c^\infty(\mathfrak{g})$ define the function $\hat{f} : \mathfrak{g} \to \mathbb{C}$ as follows:

$$\hat{f}(y) = \int_{\mathfrak{g}} \psi(B(y,x))f(x)\mathrm{d}x,$$

for $y \in \mathfrak{g}$. If d is a distribution on $\mathfrak{g}$, i.e., a linear function from $C_c^\infty(\mathfrak{g})$ to $\mathbb{C}$, define the *Fourier transform* $\hat{d}$ of d by $\hat{d}(f) = d(\hat{f})$, for $f \in C_c^\infty(\mathfrak{g})$.

For simplicity we only bother with the local summability of θ_π around the identity of G. First the problem is translated to the Lie algebra $\mathfrak{g}$ of G by the exponential map. Then it is showed that $\theta_\pi \circ \mathrm{Exp}$ is around 0 equal to a function η which Fourier transform is supported on $\mathrm{Ad}(G)\omega$, for some compact open $\omega \subset \mathfrak{g}$. Now functions of the form η are locally a linear combination of Fourier transforms of nilpotent orbital integrals. Since Fourier transforms of nilpotent orbital integrals are locally summable, so is η. Therefore θ_π is locally summable.

One of the auxiliary results for the proof of these statements is Howe's conjecture. For $\omega \subset \mathfrak{g}$ define $J(\omega)$ to be the set of G-invariant distributions with support contained in the closure of ${}^G\omega = \mathrm{Ad}(G)\omega$. For an $\mathcal{O}$-lattice L in $\mathfrak{g}$ define $J_L(\omega)$ to be the image of $J(\omega)$ in the distributions of $\mathfrak{g}/L$ under the canonical map $\phi_L : \mathfrak{g} \to \mathfrak{g}/L$.

Conjecture 2.12 (Howe)**.** *For all compact $\omega \subset \mathfrak{g}$ and all $\mathcal{O}$-lattices L of $\mathfrak{g}$:*

$$\dim J_L(\omega) < \infty.$$

This conjecture was proved by Howe in [How74] for $G = GL_n(\mathbb{F})$. Later it was proved by Harish-Chandra, see [HC99], for general G in the case that $\mathrm{char}\,\mathbb{F} = 0$.

In Chapter 4 we will study the geometry of the nilpotent orbits in $\mathfrak{g}$ and Howe's conjecture.

2.5 Bruhat-Tits building

In this section we construct the reduced and the extended building for a reductive p-adic group. These buildings are spaces on which the reductive p-adic group acts. Moreover, every point in the building gives rise to a compact subgroup of G and every compact subgroup of G fixes a point in the building.
For any reductive p-adic group Bruhat and Tits constructed a reduced and an extended building in [BT72, BT84, Tit79]. We use the notation of [MS12]. Let S be a maximal split torus of G and let S_Δ be the maximal split torus contained in the center of G. The construction of the building goes as follows:

1. We construct the standard apartment: a vector space $\mathbb{A}$ with an $N_G(S)$-action.
2. We define for each vector $x \in \mathbb{A}$ a subgroup $U_x < G$.

3. The building will be $\mathcal{B}(G) := G \times \mathbb{A}/\sim$, where $\sim$ is the equivalence relation defined by:
$$(g,x) \sim (h,y) \Leftrightarrow \exists n \in N_G(S)\ [nx = y \text{ and } g^{-1}hn \in U_x].$$

Let $\mathbb{A}_e := X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{A}_a := (X_*(S)/X_*(S_\Delta)) \otimes_{\mathbb{Z}} \mathbb{R}$.
Define $\nu : Z_G(S) \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ by:

$$\langle \nu(z), \chi|_S \rangle = -\nu(\chi(z))$$

for all $\chi \in X^*(Z_G(S))$. Let $z \in Z_G(S)$ act on $\mathbb{A}_e$ by $x \mapsto x + \nu(z)$. This action can now be extended to $N_G(S)$, $\nu : N_G(S) \to \mathrm{Aff}(\mathbb{A}_e)$.

The *standard apartment* of the extended building is $\mathbb{A}_e$, and $\mathbb{A}_a$ is the *standard apartment* of the reduced building. Define the linear map $\phi : \mathbb{A}_e \to \mathbb{A}_a$ by extending the map $X_*(S) \to X_*(S)/X_*(S_\Delta)$. So ϕ is surjective. The action of $N_G(S)$ on $\mathbb{A}_e$ gives an action on $\mathbb{A}_a$ via ϕ:
$$n\phi(x) := \phi(nx), \quad \text{for } x \in \mathbb{A}_e,\ n \in N_G(S).$$
For $\alpha \in \Phi$, $x \in \mathbb{A}_a$ and $y \in \mathbb{A}_e$, define

$$\begin{aligned}\alpha(y) &:= \langle \alpha, y \rangle, \\ \alpha(x) &:= \alpha(z),\end{aligned}$$

where z is any element in $\phi^{-1}(x)$.

Now we continue by defining the subgroups U_x. Following [Tit79] we construct subgroups $U_{\alpha,r}$ for $\alpha \in \Phi$ and $r \in \mathbb{R}$. Let r_α be the reflection associated to α. Let $u \in U_\alpha - \{1\}$, then
$$U_{-\alpha} u U_{-\alpha} \cap N_G(S) = \{m(u)\}.$$
Define $r(u) = \nu(m(u))$. The affine action $r(u)$ is an affine reflection which vector part is r_α. Let $a(\alpha, u)$ denote the affine function on $\mathbb{A}_a$ which vector part is α and which vanishing hyperplane is the fixed point set of $r(u)$. We define

$$U_{\alpha,r} := \{u \in U_\alpha \mid u = 1 \text{ or } a(\alpha, u) \geq \alpha + r\}.$$

In [MS12, §3] a more concrete description of the groups $U_{\alpha,r}$ is given: Let $\mathbb{E}$ be a field extension of $\mathbb{F}$ such that $\mathcal{G}$ is $\mathbb{E}$-split. Extend the valuation v of $\mathbb{F}$ to $\mathbb{E}$. Let $\mathcal{T}$ be a maximal $\mathbb{E}$-split torus that contains $\mathcal{S}$. Define $\Phi_{\mathcal{T}} := R(\mathcal{G}, \mathcal{T})$. Choose a Chevalley basis on $\mathfrak{g}(\mathbb{E})$, the Lie algebra of $\mathcal{G}(\mathbb{E})$. Such a basis gives rise to an isomorphism $u_\beta : \mathbb{E} \to \mathcal{U}_\beta(\mathbb{E})$ for all $\beta \in \Phi_{\mathcal{T}}$. Define $U_{\beta,r} := u_\beta(\nu^{-1}([r, \infty)))$ for $\beta \in \Phi_{\mathcal{T}}$. Let $\rho : \Phi_{\mathcal{T}} \to \Phi$ be the surjection defined by restriction of the character of $\mathcal{T}$ to $\mathcal{S}$. For $\alpha \in \Phi^{\mathrm{red}}$ and $r \in \mathbb{R}$ define

$$\begin{aligned}U_{\alpha,r} &:= U_\alpha \cap \left(\prod_{\beta \in \rho^{-1}(\alpha)} U_{\beta,r} \times \prod_{\beta \in \rho^{-1}(2\alpha)} U_{\beta,2r} \right), \\ U_{2\alpha,r} &:= U_{2\alpha} \cap U_{\alpha,r/2}.\end{aligned}$$

For $u_\alpha \in U_\alpha$ we define $\nu(u_\alpha) := \max\{r \in \mathbb{R} \mid u_\alpha \in U_{\alpha,r}\}$.
Now U_x, for $x \in \mathbb{A}_a$ or $x \in \mathbb{A}_e$, is the subgroup generated by $\bigcup_{\alpha \in \Phi} U_{\alpha,\langle x,-\alpha\rangle}$.

As announced $\mathcal{B}_a(G) := G \times \mathbb{A}_a/\sim$ and $\mathcal{B}_e(G) := G \times \mathbb{A}_e/\sim$.
The equivalence relation $\sim$ for $\mathcal{B}_a(G)$ and $\mathcal{B}_e(G)$ is:
$(g, x) \sim (h, y)$ if and only if there is a $n \in N_G(S)$ such that $nx = y$ and $g^{-1}hn \in U_x$.

If $\Omega \subset \mathbb{A}_a$ or $\Omega \subset \mathbb{A}_e$ we define

$$f_\Omega : \Phi \to \mathbb{R} \cup \{\infty\}, \quad f_\Omega(\alpha) := \sup_{x \in \Omega} \langle x, -\alpha\rangle .$$

This gives rise to the following subgroups of G:

$$\begin{aligned} U_\Omega &:= \text{ the subgroup generated by } \bigcup_{\alpha \in \Phi^{red}} U_{\alpha, f_\Omega(\alpha)}, \\ N_\Omega &:= \{n \in N_G(S) \mid nx = x \text{ for all } x \in \Omega\}, \\ P_\Omega &:= N_\Omega U_\Omega = U_\Omega N_\Omega. \end{aligned}$$

The group P_Ω is the pointwise stabilizer of Ω.
If we drop G from the notation of the building, it should be clear from the context for which group G the building is: so $\mathcal{B}_a = \mathcal{B}_a(G)$ and $\mathcal{B}_e = \mathcal{B}_e(G)$.

Now we extend $\phi : \mathbb{A}_e \to \mathbb{A}_a$ to a function $\mathcal{B}_e \to \mathcal{B}_a$ which we also denote by ϕ. So $\phi(g, x) = (g, \phi(x))$. The function ϕ is G-invariant and surjective.

Let $Y := X_*(S_\Delta) \otimes \mathbb{R} = X_*(Z(G)) \otimes \mathbb{R}$. Define π_Y and $\pi_{\mathbb{A}_a}$ to be the projections from $\mathbb{A}_a \oplus Y$ to Y and $\mathbb{A}_a$ respectively. Now we have a canonical bijection $\Pi : \mathbb{A}_e \to \mathbb{A}_a \oplus Y$, such that $\phi = \pi_{\mathbb{A}_e}\Pi$. For $x \in \mathbb{A}_a$ and $y \in Y$ we have $x \oplus y \in \mathbb{A}_a \oplus Y$. We write $(x, y) := \Pi^{-1}(x \oplus y)$ or $x + y = \Pi^{-1}(x \oplus y)$.

For $\alpha \in \Phi$ define n_α to be the smallest $r \in \mathbb{R}^+$ such that $U_{\alpha,r} \neq U_{\alpha,r+}$. For $r \in \mathbb{R}$ define the α-ceiling as: $\lceil r \rceil_\alpha := \min\{z \in n_\alpha\mathbb{Z} \mid z \geq r\}$. Let $\mathbb{A}$ be equal to $\mathbb{A}_e$ or $\mathbb{A}_a$. The affine hyperplanes

$$\mathbb{A}_{\alpha,k} := \{x \in \mathbb{A} \mid \langle x, \alpha\rangle = k\}, \text{ for } \alpha \in \Phi \text{ and } k \in n_\alpha\mathbb{Z},$$

turn $\mathbb{A}_a$ into a polysimplicial complex. An element $x \in \mathbb{A}_a$ is a *vertex* if it is the only element of an intersection of such affine hyperplanes. The polysimplicial vertices in $\mathbb{A}_e$ are $(\dim Z(G))$-dimensional hyperplanes. We call $x \in \mathbb{A}_e$ a *vertex* if it is an element of a polysimplicial vertex of $\mathbb{A}_e$. An element $x \in \mathbb{A}_e$ is a vertex if and only if $\phi(x)$ is a vertex.

For each $\Omega \subset \mathcal{B}_a$ that is contained in an apartment and each $e \in \mathbb{R}_{\geq 0}$, Schneider and Stuhler defined a group U^e_Ω in [SS97]. This group has the following properties.
For a point x, a polysimplex σ and a general subset Ω of a apartment, the following hold:

1. U^e_Ω is open if Ω is bounded.

2. U^e_Ω is compact and normal in P_Ω.

3. If $e \in \mathbb{Z}_{\geq 0}$ and x is in the interior of σ, then $U_x^e = U_\sigma^e$.

4. $U_\Omega^e \subset U_\Omega^{e'}$ whenever $e \geq e'$.

5. U_σ^e for $e \in \mathbb{N}$ form a neighborhood basis of 1 in G.

This is a part of [MS12, Theorem 5.5].
By definition a smooth representation has level greater or equal to $e \in \mathbb{R}_{\geq 0}$ if $V = \sum_{x \in \mathcal{B}_a} V^{U_x^e}$.

Define the bilinear symmetric form $\langle\ ,\ \rangle$ on $\mathbb{A}_a$ as follows:

$$\langle v, w \rangle = \sum_{\alpha \in R^+} \alpha(v)\alpha(w).$$

This form is W-invariant, because $s_\alpha R^+ = \{-\alpha\} \cup R^+ \backslash \{\alpha\}$ for all $\alpha \in \Delta$. Let $\Delta^\vee$ be the dual basis of Δ in $\mathbb{A}_a$. Let $v \in \mathbb{A}_a - \{0\}$. If $v = \sum_{\alpha^\vee \Delta^\vee} c_{\alpha^\vee} \alpha^\vee$ with $c_{\alpha^\vee} \geq 0$, then $\langle v, v \rangle > 0$. Since the form is W-invariant, it is positive definite. Thus $\langle\ ,\ \rangle$ is a W-invariant, positive definite, symmetric, bilinear form.

Choose on $\mathbb{A}_e = \mathbb{A}_a \oplus Y$ a W-invariant inner product, such that restricted to $\mathbb{A}_a$ it is equal to $\langle\ ,\ \rangle$ and $\mathbb{A}_a \perp Y$. Such an inner product exists, because $\mathbb{A}_a$ and Y are W-invariant subspaces. This inner product gives rise to a G-invariant metric d on $\mathcal{B}_e$. Since $\mathbb{A}_a \perp Y$ one has

$$d(x + y, x' + y') = (d(x, x')^2 + d(y, y')^2)^{\frac{1}{2}},$$

for $x, x' \in \mathbb{A}_a$ and $y, y' \in Y$.

2.6 Analytic groups

In this section we give an introduction to analytic groups. At the end we will show that reductive p-adic groups are analytic groups and give a proof of the Weyl integration formula. For more information on analytic groups see [Ser65]. More details on how to give algebraic groups an analytic structure are given in [PR94, §3].

Let $\mathbb{F}$ be a local non-Archimedean field. For the definition of analytic manifolds and the general objects: products, tangent spaces, etc., see [Ser65, LG3]. Our analytic manifolds are over $\mathbb{F}$.

An *analytic group* is an analytic manifold G which is also a group, such that the multiplication and inversion are analytic functions.

Example 2.13. *The group* $(\mathbb{F}, +)$ *is an analytic group.*
The group $(\mathbb{F}^\times, \cdot)$ *is an analytic group. The inverse function is, in a neighborhood of* 1, *given by:*

$$1 - X \mapsto 1 + X + X^2 + X^3 + \cdots,$$

which is analytic on $\varpi\mathcal{O}$.

For integration on the group G we refer to [PR94, §3.5]. For $g \in G$, let $L_g : G \to G$ be the function defined by left multiplication by g: $x \mapsto gx$. A G-invariant measure on G can be defined as follows: Define $n := \dim G$. Let $\omega \in \bigwedge^n T_e^*(G)$ with $\omega \neq 0$, then define $\omega_g := L_{g^{-1}}^* \omega$. The map $g \mapsto \omega_g$ is a G-invariant n-form on G and leads to a G-invariant measure.

Let H be a subgroup and an analytic submanifold of G. There is a unique analytic structure on the quotient G/H making $\pi : G \to G/H$ into a submersion by [Ser65, LG §4.5]. Now G is a so-called right principal H-bundle over the base G/H, [Ser65, LG §4.5, Theorem 6]. In particular, there is, for every $b \in G/H$, an open set U_b and an analytic isomorphism $\tau : \pi^{-1}U_b \to U_b \times H$, such that $\tau(x) = (\pi(x), \phi(x))$. The function τ is called a local trivialization.

Theorem 2.14 ([PR94])**.** *The $\mathbb{F}$-points of a linear algebraic groups defined over $\mathbb{F}$ form an $\mathbb{F}$-analytic manifold.*

We will not give a proof of this Theorem. We only describe a way to construct an analytic manifold from a linear algebraic group.
Let G be the $\mathbb{F}$-points of a linear algebraic group. Let τ be the coarsest topology on G such that every algebraic function from G to $(\mathbb{F}, |\cdot|)$ is continuous. Now we define an analytic structure on G, τ.

Let $x \in G$. Take $f_1, \dots, f_n \in \mathbb{F}[G]$ such that $T_x^*(G)$ has basis $df_1, \dots, df_n$ (such f_i exist because G is smooth). Then the map $g \mapsto (f_1(g), \dots, f_n(g))$ defines locally a chart on G. The collection of charts constructed in this way for every $x \in G$ is an atlas defining the manifold structure on G.

Example 2.15. *In this example we construct a chart around the identity of the following torus defined over $\mathbb{F}$, with char $\mathbb{F} = 2$:*

$$T := \left\{ \begin{pmatrix} e & \varpi g \\ g & e + \varpi^2 g \end{pmatrix} \,\middle|\, e^2 + \varpi^2 ge + \varpi g^2 = 1 \right\}.$$

Let $f(e,g) := e^2 + \varpi^2 ge + \varpi g^2$. Since $\frac{\partial f}{\partial g}(1,0) = \varpi^2 \neq 0$, there exists by the implicit function theorem a formal power series ϕ such that:

$$(1+X)^2 + \varpi^2 \phi(X)(1+X) + \varpi \phi(X)^2 = 1,$$

for all X in the ball of convergence of ϕ. The formal power series is $\phi(X) = \sum_{n=2}^{\infty} c_n X_n$, where $c_2 = \frac{1}{\varpi^2}$ and $c_n = c_{n-1} + \frac{1}{\varpi} c_{\frac{n}{2}}$ (for n odd, $c_{\frac{n}{2}} := 0$). Therefore, ϕ converges if $X \in \varpi\mathcal{O}$. Thus on the open set

$$U := \left\{ \begin{pmatrix} e & \varpi g \\ g & e + \varpi^2 g \end{pmatrix} \in T \,\middle|\, e - 1 \in \varpi\mathcal{O} \right\},$$

the map $c : \begin{pmatrix} e & \varpi g \\ g & e + \varpi^2 g \end{pmatrix} \mapsto e - 1$ has the analytic inverse

$$y \mapsto \begin{pmatrix} 1+y & \varpi\phi(y) \\ \phi(y) & 1 + y + \varpi^2 \phi(y) \end{pmatrix}.$$

Thus (U, c) is a chart on T.

Theorem 2.16 (Weyl Integration Formula)**.** *Let G be a reductive p-adic group, T a maximal torus of G and $W = N_G(T)/T$ its Weyl group. Assume that the measures on G, T and G/T are such that for all $f \in C_c^\infty(G)$:*

$$\int_G f(g)\mathrm{d}g = \int_{G/T}\int_T f(gt)\mathrm{d}t\mathrm{d}g.$$

Let $f \in C_c^\infty(G)$, then

$$\int_{G_T} f(g)\mathrm{d}g = \frac{1}{|W|}\int_T |D(t)| \int_{G/T} f(gtg^{-1})\mathrm{d}g\mathrm{d}t.$$

The Weyl integration formula is a well-known result in the theory of reductive groups. However, the author could not find a "spelled out" proof of the formula for the non-Archimedean case in the literature. Harish-Chandra [HC70, Lemma 42] mentions that (in the characteristic 0 case) the proof is the same as in the real case. The proof in the real case depends on the substitution rule from the theory of analytic manifolds.

As shown in [DK00, §3.13] we can choose the differential forms on G, G/T and T in such a way that

$$\int_G f(g)\mathrm{d}g = \int_T\int_{G/T} f(gt)\mathrm{d}g\mathrm{d}t.$$

If $\tau : \pi^{-1}U \to T \times U$ is a local trialization, then

$$\int_{\pi^{-1}U} f(g)\mathrm{d}g = \int_U\int_T f(\tau^{-1}(u,t))\mathrm{d}g\mathrm{d}t.$$

Proof of Theorem 2.16. The proof of [DK00, Theorem 3.14.1] in the real compact case works in this case as well. We only take a different definition of the subspace $\mathfrak{q}$. It has to be an $\mathrm{Ad}(T)$-stable $\mathbb{F}$-linear subspace of $\mathfrak{g}$ which is complementary to $\mathfrak{t}$. When the characteristic of $\mathbb{F}$ is zero, the resulting subspace is the same.

The Lie algebra $\mathfrak{t}$ of T has a complementary $\mathrm{Ad}(T)$-invariant space $\mathfrak{q}$ defined over $\mathbb{F}$. We define $\mathfrak{q} := \bigoplus_{\alpha\in R(G,T)} \mathfrak{g}_\alpha$. If T is $\mathbb{F}$-split, then clearly $\mathfrak{g}_\alpha$ is defined over $\mathbb{F}$ and hence also $\mathfrak{q}$ is defined over $\mathbb{F}$. Take $\mathbb{E}$ a Galois extension of $\mathbb{F}$ such that T is $\mathbb{E}$-split. Let $\Gamma := \mathrm{Gal}(\mathbb{E} : \mathbb{F})$. Since T is $\mathbb{E}$-split, $\mathfrak{q}$ and $\mathfrak{g}_\alpha$ are defined over $\mathbb{E}$. Thus $\mathfrak{q}$ is defined over $\mathbb{F}$ if and only if it is Γ-invariant. Let $x \in \mathfrak{g}_\alpha$, then for all $\gamma \in \Gamma$:

$$t\gamma(x)t^{-1} = \gamma(\gamma^{-1}(t)x\gamma^{-1}(t^{-1})) = \gamma(\alpha(\gamma^{-1}(t))x) = \gamma(\alpha(\gamma^{-1}(t)))\gamma(x).$$

Thus $\gamma(\alpha(\gamma^{-1}(\cdot))) \in R(G,T)$, hence $\gamma(x) \in \mathfrak{q}$. $\square$

Chapter 3

The Building and the Characters

This chapter is based on [Wit15].

Abstract

In this chapter we study the complex representations of reductive groups over local non-Archimedean fields. We use the building of the reductive group to give upper bounds for the absolute value of the character of an admissible representation and for the Weyl integration formula for certain regular elements. The upper bound for the character of a representation is based on the alternative description, depending on the building, of the character as given by R. Meyer and M. Solleveld [MS12]. Once the character and the Weyl integration formula are related to the building, the upper bounds will follow from a similar argument. Both upper bounds generalize the upper bounds given by Harish-Chandra [HC70] to groups defined over fields of positive characteristic. Finally, following Harish-Chandra's method we combine both upper bounds to show that for a maximal torus T containing a maximal split torus the character is locally summable on ${}^{G}T$.

3.1 Introduction

Let $\mathbb{F}$ be a non-Archimedean local field with characteristic p and residue field of order q. Let G be a reductive group over $\mathbb{F}$. Let π be a complex admissible representation of G. Let θ be the character of the representation π.

Conjecture 3.1. *θ is locally integrable on G.*

In the case that $\mathbb{F}$ has characteristic 0 this conjecture has been proven by Harish-Chandra, see [HC99]. He transports the problem to the Lie algebra with the exponential map. On the Lie algebra he shows that θ can locally be written as a linear combination of Fourier transforms of nilpotent orbital integrals. Since the Fourier transforms of nilpotent orbital integrals are locally summable, that completes the proof. Up to and including the moment of writing this chapter the author has not been aware of a proof of the conjecture for general $\mathbb{F}$ and G. There has been progress on proving the conjecture in two directions. It has been shown that the conjecture is true for particular groups, e.g., SL_n [Lem96] and GL_n [Lem05, Rod85]. Also for every group G defined over $\mathbb{Z}$

there is an N such that if $p > N$, then the conjecture holds [CGH14]. In both [Rod85] and [CGH14] one more or less generalizes the proof given by Harish-Chandra to fields of positive characteristic. One follows the proof of Harish-Chandra to show that the trace is a linear combination of Fourier transforms of nilpotent orbital integrals, to prove the conjecture when the characteristic is large enough. For each step in the proof one tries to generalize this step to positive characteristic and/or keep track of the assumptions made, see for example [DeB02a]. That the nilpotent distributions are locally summable in large positive characteristic is shown by motivic integration in [CGH14]. Here one shows that θ is locally summable in characteristic 0 if and only if it is locally summable for all large p.

If π is a cuspidal representation Harish-Chandra proves (in characteristic 0) that θ is locally summable in another way, see [HC70]. His proof consists of four steps:

1. For every $g \in G$ there exist a compact neighborhood ω of g, a $C \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$ such that for all $\gamma \in \omega$
$$|\theta(\gamma)| \leq C|\lambda(\gamma)|^n |D(\gamma)|^{-\frac{1}{2}}.$$

2. For all $\epsilon \geq 0$
$$\int_{G_T} |D(x)|^{-\frac{1}{2}-\epsilon} f(x)\mathrm{d}x = |N_G(T)/T|^{-1} \int_T |D(t)| \int_{G/T} |D(t)|^{-\frac{1}{2}-\epsilon} f(gtg^{-1})\mathrm{d}g\mathrm{d}t.$$

3. For every $f \in C_c^\infty(G)$ there exists a $C \in \mathbb{R}_{>0}$ such that for all regular $t \in T$
$$\int_{G/T} f(gtg^{-1})\mathrm{d}g \leq C|D(t)|^{-\frac{1}{2}}.$$

4. For small $\epsilon \geq 0$ the function $|\lambda(t)|^n |D(t)|^{-\epsilon}$ is locally summable on T.

The local summability of the character on G follows from these four statements, because, when the characteristic is 0, there are only finitely many conjugacy classes of maximal $\mathbb{F}$-tori:

$$\begin{aligned} \int_{G_T} |f\theta|\mathrm{d}g &\leq \int_{G_T} C_\theta |\lambda(g)|^n |D(g)|^{-\frac{1}{2}} |f(x)|\mathrm{d}x \\ &= |W|^{-1} \int_T c_\theta |D(t)|^{\frac{1}{2}} \int_{G_T} |\lambda(gtg^{-1})|^n |f(gtg^{-1})|\mathrm{d}g\mathrm{d}t \\ &\leq \frac{C_\theta C_f}{|W|} \int_T |\lambda(t)|^n \mathrm{d}t \leq C_\epsilon \int_T |D(t)|^{-\epsilon}\mathrm{d}t. \end{aligned} \tag{3.1}$$

In this chapter we give similar estimates as in statements 1 and 3 in the case that $\gamma \in Z_G(S)$, and we prove statement 2 and 4. The advantage of our method is that it also works in positive characteristic. To be more precise we will prove the following:

Theorem 3.2. *For all maximal tori T of G:*

1. *For all $\epsilon \geq 0$*
$$\int_{G_T} |D(x)|^{-\frac{1}{2}-\epsilon} f(x)\mathrm{d}x = |N_G(T)/T|^{-1} \int_T |D(t)| \int_{G/T} |D(t)|^{-\frac{1}{2}-\epsilon} f(gtg^{-1})\mathrm{d}g\mathrm{d}t.$$

2. *For small $\epsilon > 0$ the function $sd(t)^n|D(t)|^{-\epsilon}$ is locally summable on T.*

Let S be a maximal $\mathbb{F}$-split torus and Φ the roots of S and G.

3. *For every $g \in G$ there exist a compact neighborhood ω of g, a $C \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$ such that for all regular $\gamma \in \omega \cap {}^G Z_G(S)$*

$$|\theta(\gamma)| \leq C(ht(\Phi)sd(\gamma))^n |D(\gamma)|^{-\frac{1}{2}}.$$

If moreover $T \subset Z_G(S)$, then

4. *For every $f \in C_c^\infty(G)$ there exists a $C \in \mathbb{R}_{>0}$ such that for all regular $t \in T$*

$$\int_{G/T} f(gtg^{-1})\mathrm{d}g \leq C|D(t)|^{-\frac{1}{2}}.$$

The first statement follows directly from the Weyl integration formula, Theorem 2.16. As the calculation (3.1) shows, we get the following theorem as consequence.

Theorem 3.3. *Let (ρ, V) be a G-representation of finite length with character θ and $f \in C_c^\infty(G)$, then for every torus T containing a maximal $\mathbb{F}$-split torus:*

$$\int_{{}^G T} f(g)\theta(g)\mathrm{d}g < \infty.$$

Assume that $\gamma \in Z_G(S)$ is compact. We use an alternative description of the character, which uses the reduced Bruhat-Tits building of the reductive group G, given by Meyer and Solleveld in [MS10] and [MS12], for the local upper bound of the character. The non-compact case is deduced from the compact case via Casselman's method and the displacement function. For the upper bound of the Weyl integral $\int_{G/T} f(g\gamma g^{-1})\mathrm{d}g$ we use the extended and the reduced buildings. Both estimates are related to the fixed points of γ in a reduced building.

After introducing and recalling some notations, we study the distribution of γ-fixed points in the reduced building. Then we give an upper bound for the trace of a representation with finite level. After that section we give an upper bound for $\int_{G/T} f(gtg^{-1})\mathrm{d}g$. Then we combine both upper bounds to a proof of the local summability of θ on $\{gtg^{-1} : g \in G, t \in T\}$. At the end of this chapter we look at GL_2 and give a direction for further research.

Most of the lemmas and theorems about the fixed points in the building and the relation between the Weyl integral and the fixed points are inspired by examples such as $SL_2(\mathbb{F})$ and $GL_3(\mathbb{F})$.

3.2 Notations

Let $\mathbb{F}$ be a non-archimedean local field with valuation $\nu : \mathbb{F}^\times \to \mathbb{R}$, ring of integers $\mathcal{O}$ and uniformizer ϖ. Define q to be the order of the residue field of $\mathbb{F}$. Let p be the

characteristic of $\mathbb{F}$. Let k be an algebraic closure of $\mathbb{F}$.
$\mathcal{G}, \mathcal{S}, \mathcal{T}, \mathcal{U}$ are linear algebraic groups over $\mathbb{F}$ and G, S, T, U are the $\mathbb{F}$-points of these groups, respectively. The Lie algebra of a group G is denoted by $\mathfrak{g}$. $\mathcal{G}$ is a connected reductive group and $\mathcal{T}$ is a maximal torus in $\mathcal{G}$.

Let $Z = Z(G)$ be the center of G and $Z(G)^0$ the identity component of Z.
Let S be a maximal $\mathbb{F}$-split torus of G.
Let $S_\Delta := S \cap Z(G)^0$ be the maximal $\mathbb{F}$-split torus in $Z(G)$.
The Weyl group of S is denoted by $W := N_G(S)/Z_G(S)$.
For $\omega \subset G$, define ${}^G\omega := \{gwg^{-1} : w \in \omega, g \in G\}$.
The root system of (G, S) is denoted by Φ. Let Φ^+ be a system of positive roots and Δ the simple roots of Φ^+. Define on Φ^+ the *height function* $\mathrm{ht} : \Phi^+ \to \mathbb{N}$ as usual:

$$\begin{aligned} \mathrm{ht}(\alpha) &= 1, && \text{for all } \alpha \in \Delta, \\ \mathrm{ht}(\alpha+\beta) &= \mathrm{ht}(\alpha) + \mathrm{ht}(\beta), && \text{if } \alpha, \beta, \alpha+\beta \in \Phi^+. \end{aligned}$$

Let $\mathrm{ht}(\Phi) := \max_{\alpha\in\Phi^+} \mathrm{ht}(\alpha)$. Define $U^+ := \Pi_{\alpha\in\Phi^+} U_\alpha$ and $U^- := \Pi_{\alpha\in\Phi^-} U_\alpha$. Recall that $\nu(u_\alpha) := \max\{r \in \mathbb{R} \mid u_\alpha \in U_{\alpha,r}\}$ for $u_\alpha \in U_\alpha$.

Let γ be a regular semisimple element. Let $\mathbb{E}$ be a field extension of $\mathbb{F}$ such that $T := Z_G^0(\gamma)$ is $\mathbb{E}$-split. Extend the valuation ν of $\mathbb{F}$ to $\mathbb{E}$. Let $\tilde{\Phi} := R(\mathcal{G}, \mathcal{T})$. Define the *singular depth* of γ as follows:

$$\mathrm{sd}(\gamma) := \max_{\alpha\in\tilde{\Phi}} \nu(\alpha(t) - 1).$$

3.3 γ-Fixed Points and $D(\gamma)$

An element $g \in G$ is called *compact* if and only if it is contained in a subgroup K that is compact modulo $Z(G)$.
This section gives a proof of the following theorem.

Theorem 3.4. *Let $x, y \in \mathbb{A}_a$ and let $\gamma \in Z_G(S)$ be regular and compact, then*

$$\#\{ux : u \in U^+ \cap P_y \mid \gamma ux = ux\} \leq |D(\gamma)|^{-\frac{1}{2}}.$$

In the proof of Theorem 3.4 we need Lemma 3.9. Besides some standard facts the proof of Theorem 3.4 uses only this lemma, which is trivial when G is $\mathbb{F}$-split. The main part of this section is dedicated to the proof of Lemma 3.9.

First we will discuss some consequences of the following theorem.
Define $\mathcal{G}_a := (k, +)$ and $G_a := \mathcal{G}_a(\mathbb{F})$.

Theorem 3.5. *Let $\mathcal{A}$ be an $\mathbb{F}$-split solvable group, $\mathcal{T}$ a maximal $\mathbb{F}$-torus of $\mathcal{A}$ and $\mathcal{A}_u$ the unipotent radical of $\mathcal{A}$.*

1. *There exists an $\mathbb{F}$-isomorphism of varieties $\psi : \mathcal{A}_u \to \mathcal{G}_a^n$ with $\psi(e) = 0$ and a rational representation ρ of $\mathcal{T}$ in k^n defined over $\mathbb{F}$ such that $\psi(tgt^{-1}) = \rho(t)\psi(g)$ for all $g \in \mathcal{A}$ and $t \in \mathcal{T}$.*

2. *For $x, y \in \mathcal{G}_a^n$ we have $\psi(\psi^{-1}(x)\psi^{-1}(y)) = x + y + \sum_{i\geq 2} F_i(x,y)$, where $F_i : \mathcal{G}_a^n \times \mathcal{G}_a^n \to \mathcal{G}_a^n$ is a polynomial map of degree i.*

3. *The weights of $\mathcal{T}$ for ρ are the weights of $\mathcal{T}$ in $\mathfrak{g}$.*

Proof. See Proposition 14.3.11 in [Spr98]. □

Corollary 3.6. *Let $\mathcal{S}$ be an $\mathbb{F}$-split torus and let $\mathcal{U}$ be an $\mathbb{F}$-split unipotent group with an algebraic action of $\mathcal{S}$. Let $n = \dim \mathcal{U}$. Assume that $\alpha \in X^*(\mathcal{S})$ is the only weight for $\mathcal{S}$ on $\mathfrak{u}$ and that α is non-trivial. Then there is an $\mathbb{F}$-isomorphism ψ between the groups $\mathcal{U}$ and $\mathcal{G}_a^n$ such that $\psi(sus^{-1}) = \alpha(s)\psi(u)$ for all $s \in \mathcal{S}$.*

Proof. Apply Theorem 3.5 to $\mathcal{A} = \mathcal{S} \ltimes \mathcal{U}$.
Let $\psi : \mathcal{U} \to \mathcal{G}_a^n$ be an $\mathbb{F}$-isomorphism as in Theorem 3.5. Then

$$\psi(\psi^{-1}(x)\psi^{-1}(y)) = x + y + \sum_{i\geq 2} F_i(x,y),$$

where $F_i(x,y) : \mathcal{G}_a^n \times \mathcal{G}_a^n \to \mathcal{G}_a^n$ is a polynomial map of degree i.
The weights of $\mathcal{S}$ for ρ are the weights of $\mathcal{S}$ in $\mathfrak{g}$.
Since the only weight of $\mathcal{S}$ in $\mathfrak{u}$ is α, the weight of $\mathcal{S}$ for ρ is α. Therefore, $\rho(s) = \alpha(s)$ for all $s \in \mathcal{S}$. Also,

$$\begin{aligned}\psi(s\psi^{-1}(x)\psi^{-1}(y)s^{-1}) &= \rho(s)(x + y + \sum_{i\geq 2} F_i(x,y)) \\ \psi(s\psi^{-1}(x)s^{-1}s\psi^{-1}(y)s^{-1}) &= \psi(\psi^{-1}(\rho(s)x)\psi^{-1}(\rho(s)y)) \\ &= \rho(s)x + \rho(s)y + \sum_{i\geq 2} F_i(\rho(s)x, \rho(s)y).\end{aligned}$$

Since $\mathrm{im}(\rho) \cong k^\times$ and k is infinite, $x + y + \sum_{i\geq 2} F_i(x,y)$ is a homogeneous polynomial map of degree 1. Therefore, $\psi(\psi^{-1}(x)\psi^{-1}(y)) = x + y$. So ψ is a group homomorphism between $\mathcal{U}$ and $\mathcal{G}_a^n$. □

Lemma 3.7. *Let $\mathcal{S}$ be a maximal $\mathbb{F}$-split torus of the reductive group $\mathcal{G}$. Let $\mathcal{T} \subset Z_{\mathcal{G}}(\mathcal{S})$ be a maximal $\mathbb{F}$-torus, $\alpha \in R(\mathcal{G}, \mathcal{S})$ and $\mathcal{U}_\alpha$ the unipotent group for α. There are group isomorphisms $\psi_1 : \mathcal{U}_\alpha/\mathcal{U}_{2\alpha} \to \mathcal{G}_a^m$ and $\psi_2 : \mathcal{U}_{2\alpha} \to \mathcal{G}_a^n$ such that for all $r \geq 0$:*

1. *$\psi_1(U_{\alpha,r}/U_{2\alpha,r})$ is an $\mathcal{O}$-lattice in $U_\alpha/U_{2\alpha}$,*

2. *$\psi_2(U_{2\alpha,r})$ is an $\mathcal{O}$-lattice in U_α,*

3. *The conjugation action of $\mathcal{T}$ on $\mathcal{U}_\alpha/\mathcal{U}_{2\alpha}$ (resp. $\mathcal{U}_{2\alpha}$) gives rise to a rational linear action ρ_1 (resp. ρ_2) of $\mathcal{T}$ on $\psi_1(\mathcal{U}_\alpha/\mathcal{U}_{2\alpha})$ (resp. $\psi_2(\mathcal{U}_{2\alpha})$). Moreover the weights of $\mathcal{T}$ for ρ_1 (resp. ρ_2) are the weights of $\mathcal{T}$ in $\mathfrak{u}_\alpha/\mathfrak{u}_{2\alpha}$ (resp. $\mathfrak{u}_{2\alpha}$).*

Proof. We will only consider the case with $\psi_1 : \mathcal{U}_\alpha/\mathcal{U}_{2\alpha,2r} \to \mathcal{G}_a^m$. The proof with ψ_2 goes analogously.

Let $\mathbb{E}$ be a finite field extension of $\mathbb{F}$ such that $\mathcal{T}$ is $\mathbb{E}$-split. The group $\mathcal{U}_\alpha$ is stable under conjugation with $\mathcal{T}$, because $\mathcal{T} \subset Z_{\mathcal{G}}(\mathcal{S})$. Define $\psi_{\mathcal{S}} : \mathcal{U}_\alpha/\mathcal{U}_{2\alpha} \to \mathcal{G}_a^m$ to be an

$\mathbb{F}$-group isomorphism as in Corollary 3.6. Let $\{\beta_1, \ldots, \beta_m\}$ be the subset of the roots of $\mathcal{G}$ relative to $\mathcal{T}$ such that $\beta_i|_{\mathcal{S}} = \alpha$. Define $\psi_{\mathcal{T}} : \mathcal{U}_\alpha/\mathcal{U}_{2\alpha} \to \mathcal{G}_a^m$ by its inverse: $\psi_{\mathcal{T}}^{-1}(x_1, \ldots, x_m) := \prod_{i=1}^m u_{\beta_i}(x_i) \mod \mathcal{U}_{2\alpha}$ where the $u_{\beta_i} : \mathcal{G}_a \to \mathcal{U}_\beta$ are chosen in such a way that

$$U_{\alpha,r}/U_{2\alpha,r} := \{\prod_{i=1}^m u_{\beta_i}(x_i) \mod U_{2\alpha} : \nu(x_i) \geq r\} \cap U_\alpha/U_{2\alpha}.$$

The map $\psi_{\mathcal{S}}\psi_{\mathcal{T}}^{-1} : \mathcal{G}_a^m \to \mathcal{G}_a^m$ is an $\mathbb{E}$-group isomorphism. Since $\psi_{\mathcal{S}}\psi_{\mathcal{T}}^{-1}$ preserves the action of $\mathcal{S}$, it is also an $\mathbb{E}$-linear map. Therefore, there is an $\mathbb{F}$-structure on $\mathcal{G}_a^m$ (in the sense of vector spaces) such that $\psi_{\mathcal{T}}$ is an $\mathbb{F}$-isomorphism between $\mathcal{G}_a^n(\mathbb{F})$ and U_α. The group $\psi_{\mathcal{T}}(U_{\alpha,r}(\mathbb{E})/U_{2a,r}(\mathbb{E}))$ is an $\mathcal{O}_{\mathbb{E}}$-lattice. So

$$\psi_{\mathcal{T}}(U_{\alpha,r}/U_{2\alpha,r}) = \psi_{\mathcal{T}}(U_{\alpha,r}(\mathbb{E})/U_{2\alpha,r}(\mathbb{E})) \cap \mathcal{G}_a^m(\mathbb{F})$$

is an $\mathcal{O}$-lattice.
The rank of the $\mathcal{O}$-lattice is m:
For all $x \in \mathbb{E}, \beta_i$ and $r \in \mathbb{R}$ one has

$$u_{\beta_i}(x) \in U_{\beta_i,r} \Leftrightarrow u_{\beta_i}(\varpi x) \in U_{\beta_i,r+1}.$$

Since multiplication with ϖ respects the $\mathbb{F}$-structure on $\mathcal{G}_a^m(\mathbb{F})$, one has

$$[U_{\alpha,r}U_{2\alpha}/U_{2\alpha} : U_{\alpha,r+1}U_{2\alpha}/U_{2\alpha}] = q^l,$$

where l is the rank of the $\mathcal{O}$-lattice $\psi_1(U_{\alpha,r}/U_{2\alpha})$.
For all β_i one has $\bigcup_{r\in\mathbb{R}} U_{\beta_i,r} = U_{\beta_i}$ and $U_{\beta_i,r} \leq U_{\beta_i,s}$ whenever $s \leq r$. Therefore, also $\bigcup_{r\in\mathbb{R}} U_{\alpha,r}U_{2\alpha}/U_{2\alpha} = U_\alpha/U_{2\alpha}$. Since the rank of $U_{\alpha,r}$ is the same for all $r \in \mathbb{R}$, the rank of $U_{\alpha,r}$ is m.
By construction of $\psi_{\mathcal{T}}$ the weights of ρ_1 are the same as the weights of the conjugation action of $\mathcal{T}$ on $\mathfrak{u}_\alpha/\mathfrak{u}_{2\alpha}$. □

Lemma 3.8. *Let $L' < L$ be $\mathcal{O}$-lattices in $\mathbb{F}^n$ (of rank n) and $M \in GL_n(\mathbb{F})$ such that $ML' < L'$ and $ML < L$. Let $v \in L$, then*

$$\#\{l \in L/L' : Ml + v \in L'\} \leq |\det M|^{-1}.$$

Proof. We may assume that there exists at least one $l \in L/L'$ such that $Ml + v \in L'$. Take $n \in \mathbb{N}$ such that $L < \varpi^{-n}L'$. Then

$$\begin{aligned}\#\{l \in L/L' : Ml + v \in L'\} &= \#\{l \in L/L' : Ml \in L'\} \\ &\leq \#\{l \in \varpi^{-n}L'/L' : Ml \in L'\}.\end{aligned}$$

We will now estimate the last number.
Take a basis for L'.
Let D be the Smith normal form of M with respect to L', i.e., there are $P, Q \in GL(L')$ such that $PMQ = D$ and D is a diagonal matrix. Then

$$\begin{aligned}\#\{l \in \varpi^{-n}L'/L' : Ml \in L'\} &= \#\{l \in \varpi^{-n}L'/L' : Dl \in L'\} \\ &\leq |\det D|^{-1} = |\det M|^{-1}.\end{aligned}$$

The inequality follows from the fact that for all $c \in \mathbb{F}^\times$, the number of $a \in \mathcal{O}/\varpi^n\mathcal{O}$ such that $ca \equiv 0 \mod \varpi^n$ is bounded by $q^{\nu(c)} = |c|^{-1}$. □

For $q, r \in \mathbb{R}$, define $\exp_q(r) := q^r$. Recall that T is a maximal torus of G containing a maximal split torus S.

Lemma 3.9. *Let $t \in T$ be compact. Let $r, s \in \mathbb{R}$ and $r < s$. Let V be a set of representatives for the cosets of $U_{2\alpha,s}$ in $U_{2a,r}$ and U a set of representatives for the cosets of $U_{\alpha,s}U_{2\alpha,r}$ in $U_{\alpha,r}$.*

1. *$\{uv : u \in U,\ v \in V\}$ is a set of representatives for the cosets of $U_{\alpha,s}$ in $U_{\alpha,r}$.*
2. *For $w, w' \in U_{\alpha,r}$ one has*

$$\#\{(u,v) \in U \times V \mid w'[(uv)^{-1}, t]w \in U_{\alpha,s}\} \leq \exp_q\left(\sum_{\beta \in \rho^{-1}\{\alpha, 2\alpha\}} \nu(\beta(t) - 1)\right).$$

Proof. Define $\psi : U_\alpha \to U_\alpha / U_{2\alpha}$ to be the quotient map.
We first prove the following:

$$\#\{u \in U \mid \psi(w'[u^{-1}, t]w) \in \psi(U_{\alpha,s})\} \leq \exp_q\left(\sum_{\beta \in \rho^{-1}(\alpha)} \nu(\beta(t) - 1)\right).$$

The set $\psi(U_{\alpha,s})$ is an $\mathcal{O}$-lattice and $\psi(u) \mapsto \psi([u,t])$ is a linear action on the lattice. Since this action has determinant $\prod_{\beta \in \rho^{-1}(\alpha)}(\beta(t) - 1)$, the inequality follows from Lemma 3.8.

Now we prove that for every $u \in U$:

$$\#\{v \in V \mid w'[((uv)^{-1}, t]w \in U_{\alpha,s}\} \leq \exp_q\left(\sum_{\beta \in \rho^{-1}(2\alpha)} \nu(\beta(t) - 1)\right).$$

We may assume that $\psi(w'[u^{-1}, t]w) \in \psi(U_{\alpha,s})$.
So $w'[u^{-1}, t]w = u_\alpha v'$ with $u_\alpha \in U_{\alpha,s}$ and $v' \in U_{2\alpha}$. Since $U_{2\alpha}$ is in the center of U_α and stable under conjugation with t, one has

$$w'[(uv)^{-1}, t]w = w'[u^{-1}, t]w[v^{-1}, t] = u_\alpha v'[v^{-1}, t].$$

The latter is in $U_{\alpha,s}$ if and only if $v'[v^{-1}, t] \in U_{2\alpha,s}$. So

$$\{v \in V \mid w'[((uv)^{-1}, t]w \in U_{\alpha,s}\} = \{v \in V \mid v'[v^{-1}, t] \in U_{2\alpha,s}\}.$$

One gets the upper bound for the number of v's in the last set in the same way as in the case with U.
Combining both upper bounds results in the upper bound of the Lemma. □

Proof of Theorem 3.4. Since γ is compact, it fixes $\mathbb{A}_a$ pointwise.
Let Φ_o^+ be the set of positive roots associated with U^+. Define $\Phi^+ := \{\alpha \in \Phi_o^+ \mid 2\alpha \notin \Phi_o^+\} = \{\alpha_1, \ldots, \alpha_k\}$. Write $u \in U^+ \cap P_y$ as $u = \prod_{i=1}^k u_{\alpha_i}$. Now $\gamma u x = ux$ if and only if $[u^{-1}, \gamma]_{\alpha_i} \in U_{\alpha_i, -\alpha_i(x)}$ for all i. We will count the number of fixed points in the orbit of x under $U^+ \cap P_y$.

Let R_α be a set of representatives of the cosets $U_{y,-\alpha(y)}/U_{x,-\alpha(x)}$ for each $\alpha \in \Phi^+$.
We use the following bijection between $\prod_{\alpha\in\Phi^+} R_\alpha$ and $\{ux : u \in U_y^+\}$:

$$(u_\alpha)_{\alpha\in\Phi^+} \mapsto \left(\prod_{\alpha\in\Phi^+} u_\alpha\right) x.$$

Let $v \in U_y^+$.
CLAIM: The number of $v_\beta \in U_{\beta,-\beta(y)}/U_{\beta,-\beta(x)}$ such that there is a u with $u_\alpha = v_\alpha$ for the roots α with $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$, $u_\beta = v_\beta$ and $\gamma ux = ux$ is bounded by

$$\exp_q\left(\sum_{\tilde\beta\in\rho^{-1}(\beta),\,\tilde\beta\in\rho^{-1}(2\beta)} \nu(\tilde\beta(t)-1)\right).$$

If $\gamma ux = ux$, then $[u^{-1},\gamma]_\beta \in U_{\beta,-\beta(x)}$. Now $[u^{-1},\gamma]_\beta = w[u^{-1},\gamma]w'$, where $w, w' \in U_\beta$ only depend on the u_α with $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$. Hence by Lemma 3.9 the number of v_β's is bounded by $\exp_q\left(\sum_{\tilde\beta\in\rho^{-1}(\beta),\,\tilde\beta\in\rho^{-1}(2\beta)} \nu(\tilde\beta(t)-1)\right)$.

The claim allows us to prove with induction on the height of the roots that

$$\left|\left\{u \in \prod_{\alpha\in\Phi^+} R_\alpha \;\middle|\; \gamma ux = ux\right\}\right| \le \exp_q\left(\sum_{\beta\in\rho^{-1}(2\Phi^+),\beta\in\rho^{-1}(\Phi^+)} \nu(\beta(t)-1)\right).$$

Since T/S is compact, $\nu(\alpha(t)-1) \ge 0$ for all $\alpha \in R(G,T)$ with $\alpha|_S = 0$. Thus

$$\begin{aligned}
|\{ux : u \in U^+ \cap P_y \mid \gamma ux = ux\}| &= \left|\left\{u \in \prod_{\alpha\in\Phi^+} R_\alpha \;\middle|\; \gamma ux = ux\right\}\right| \\
&\le \exp_q\left(\sum_{\beta\in\rho^{-1}(2\Phi^+),\beta\in\rho^{-1}(\Phi^+)} \nu(\beta(t)-1)\right) \le |D(\gamma)|^{-\frac{1}{2}}.
\end{aligned}$$

So $\{ux : u \in U^+ \cap P_y \mid \gamma ux = ux\}$ has at most $|D(\gamma)|^{-\frac{1}{2}}$ points. $\square$

3.4 An upper bound for the character

The first part of this section up to and including Theorem 3.12 is essentially in [MS10] and [MS12].
Let (ρ, V) be an admissible G-representation of level e.
For an open compact subgroup K of G we denote 1_K for the indicator function of K in G and $\langle K\rangle := \frac{1_K}{\mathrm{vol}(K)}$.
Let $\mathcal{B}$ be the reduced building of G and let $\mathbb{A}$ be the standard apartment of S in $\mathcal{B}$. Define O to be the origin of $\mathbb{A}$. For a finite subcomplex $\Sigma \subset \mathcal{B}$ and $g \in G$ define

$$u_\Sigma^e := \sum_{\sigma\in\Sigma} (-1)^{\dim\sigma} \langle U_\sigma^e\rangle,$$

$$\tau_\Sigma(g) := \sum_{\sigma\in\Sigma^g} (-1)^{\dim\sigma} \epsilon_\sigma(g) \mathrm{tr}(\rho(g), V^{U_\sigma^e}),$$

where Σ^g is the set of g-stable polysimplices in Σ and $\epsilon_\sigma(g)$ is 1 if g preserves the orientation of σ and -1 otherwise. For $r \in \mathbb{R}$ define

$$\begin{aligned}\mathbb{A}_r^{\alpha+} &:= \{x \in \mathbb{A} \mid \alpha(x) > r\},\\ \mathbb{A}_r^{\alpha 0} &:= \{x \in \mathbb{A} \mid \alpha(x) \in [-r, r]\},\\ \mathbb{A}_r^{\alpha-} &:= \{x \in \mathbb{A} \mid \alpha(x) < -r\}.\end{aligned}$$

For any map $\epsilon : \Phi \to \{+, 0, -\}$ we write

$$\mathbb{A}_r^\epsilon := \bigcap_{\alpha \in \Phi} \mathbb{A}_r^{\alpha\, \epsilon(\alpha)}.$$

Let $\mathbb{A}_r^b$ be the union of the bounded $\mathbb{A}_r^\epsilon$. Define $\mathcal{B}_r := P_{\mathsf{O}} \mathbb{A}_r^b$.

Lemma 3.10. *[MS12, Lemma 8.2] Let $r \in \mathbb{Z}_{\geq e}$ and let $\Sigma \subset \mathcal{B}$ be any finite convex subcomplex that contains $\mathcal{B}_{r-e}$. Then*

$$\langle U_{\mathsf{O}}^r \rangle\, u_\Sigma^e = \langle U_{\mathsf{O}}^r \rangle\, u_{\mathcal{B}_{r-e}}^e.$$

Take r such that $\mathbb{A}_1^b \subset \mathbb{A}^{U_{\mathsf{O}}^r}$. For $n \in \mathbb{N}_{\geq 1}$, then $\mathbb{A}_n^b \subset \mathbb{A}^{U_{\mathsf{O}}^{nr}}$. Define $C_n := P_{\mathsf{O}} \mathbb{A}^{U_{\mathsf{O}}^{nr}} = \mathcal{B}^{U_{\mathsf{O}}^{nr}}$. Now C_n is a finite P_{O}-invariant convex subcomplex containing $\mathcal{B}_n$.

For $\Sigma \subset \mathcal{B}$ define $\Sigma^0 := \{v \in \Sigma \mid v \text{ is a vertex of } \mathcal{B}\}$.

Theorem 3.11. *For each $f \in C_c^\infty(P_{\mathsf{O}})$ and finite P_{O}-invariant convex subcomplex Σ_0 such that* $\operatorname{im} f \subset \sum_{x \in \Sigma_0^0} V^{U_x^e}$ *one has*

$$\operatorname{tr}(f, V) = \int_{P_{\mathsf{O}}} f(g) \tau_{\Sigma_0}(g) \mathrm{d}g.$$

Proof. See the proofs in [MS10, Theorem 2.4 and Proposition 4.1]. ☐

Theorem 3.12. *If $\gamma \in Z_G(S)$ is regular semisimple, then*

1. tr *is constant on* $\gamma U_{\mathsf{O}}^{\max(\mathrm{sd}(\gamma), e)}$,
2. *for all $\Sigma \subset \mathcal{B}_a$, τ_Σ is constant on* $U_x^{\max(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma), e)}\gamma$.

Proof. See the proof of [MS12, Theorem 7.2]. ☐

Corollary 3.13. *Let $\gamma \in P_{\mathsf{O}}$ and $r \geq \mathrm{ht}(\Phi)\mathrm{sd}(\gamma)$, then*

$$\operatorname{tr}_\rho(\gamma, V) = \tau_{C_{r-e}}(\gamma).$$

Proof. Since $\gamma \in P_{\mathsf{O}}$ and $U_{\mathsf{O}}^r \lhd P_{\mathsf{O}}$, the endomorphisms $\rho(\gamma)$ and $\rho(\langle U_{\mathsf{O}}^r \rangle)$ commute. Thus $\operatorname{im} \rho(\gamma \langle U_{\mathsf{O}}^r \rangle) \subset V^{U_{\mathsf{O}}^r}$. There exists a finite convex subcomplex Σ containing $\mathcal{B}_{r-e}$ such that $\langle U_{\mathsf{O}}^r \rangle u_\Sigma^e V = V^{U_{\mathsf{O}}^r}$, because U_{O}^r is an open compact group and V is admissible. So $\langle U_{\mathsf{O}}^r \rangle u_{\mathcal{B}_{r-e}}^e V = V^{U_{\mathsf{O}}^r}$ by Lemma 3.10. Since $\mathcal{B}_{r-e}$ is P_{O}-invariant, the space $u_{\mathcal{B}_{r-e}}^e V$ is U_{O}^r-invariant. Thus $V^{U_{\mathsf{O}}^r} \subset u_{\mathcal{B}_{r-e}}^e V$. Now C_{r-e} is convex and contains $\mathcal{B}_{r-e}$, so the requirements in Theorem 3.11 are fulfilled for $f = \gamma \langle U_{\mathsf{O}}^r \rangle$ and $\Sigma_0 = C_{r-e}$. Therefore, by Theorem 3.12,

$$\operatorname{tr}(\gamma, V) = \operatorname{tr}(\gamma \langle U_{\mathsf{O}}^r \rangle V) = \frac{1}{\mathrm{vol}(U_{\mathsf{O}}^r)} \int_{U_{\mathsf{O}}^r} \tau_{C_{r-e}}(\gamma g) \mathrm{d}g = \tau_{C_{r-e}}(\gamma).$$ ☐

Lemma 3.14. *Let $h \in P_x$. There exists a C such that for all $g \in hU_x^0$ and all simplices $\sigma \in \mathcal{B}^g$:*

$$|\mathrm{tr}(\rho(g), V^{U_\sigma^e})| \leq C.$$

Proof. Denote $Z(G)$ with Z. Let N be the order of the quotient group $P_x/(ZU_x^0)$. Take $z \in Z$ and $k \in U_x^0$ such that $h^N = zk$. Define $k' := g^N z^{-1}$, then $k' \in U_x^0$ and $g^N = zk'$.

Since g and z fix σ, so does k'. Hence k' is in $U_x^0 \cap P_\sigma$. Let $m := \dim V^{U_\sigma^e}$. Choose on $V^{U_\sigma^e}$ a basis such that $\rho(z)$ and $\rho(g)$ are upper triangular matrices. Now also $\rho(k')$ is an upper triangular matrix. Let $\kappa_1, \ldots, \kappa_m$, $\lambda_1, \ldots, \lambda_m$ and $\nu_1, \ldots, \nu_m$ be the entries on the diagonal of the matrices $\rho(g)$, $\rho(z)$ and $\rho(k')$, respectively. Define $c(z, \sigma) := \sum_{i=1}^m |\lambda_i|^{\frac{1}{N}}$. Since k' is contained in a compact subgroup acting on $V^{U_\sigma^e}$, $|\nu_i| = 1$. Thus $|\kappa_i^N| = |\lambda_i \nu_i| = |\lambda_i|$. Hence $|\mathrm{tr}(\rho(g), V^{U_\sigma^e})| \leq \sum_{i=1}^m |\lambda_i|^{\frac{1}{N}} = c(z, \sigma)$.
Since z is in the center of G, for all σ and σ' in the same G-orbit, $c(z, \sigma) = c(z, \sigma')$. (The eigenvalues and their multiplicity for $\rho(z)$ on $V^{U_\sigma^e}$ and $V^{U_{\sigma'}^e}$ are the same.) Since there are only finitely many G-orbits of simplices in $\mathcal{B}$, there is a C_z such that $c(z, \sigma) \leq C_z$ for all simplices $\sigma \in \mathcal{B}$. Thus $|\mathrm{tr}(\rho(g), V^{U_\sigma^e})| \leq C_z$ for all $\sigma \in \mathcal{B}^g$. $\square$

Recall that for regular semisimple elements γ,

$$D(\gamma) := \prod_{\alpha \in R(G, Z_G^0(\gamma))} (\alpha(\gamma) - 1).$$

Define $n := \dim \mathbb{A}_a$.

Proposition 3.15. *Let $g \in P_x$. There exists a $C \in \mathbb{R}$ depending only on the affine building of G, the element g and the representation (ρ, V), such that for all semisimple regular $\gamma \in {}^G Z_G(S) \cap gU_x^0$:*

$$|\mathrm{tr}(\gamma, V)| \leq C(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma) + 1)^n |D(\gamma)|^{-\frac{1}{2}}.$$

Proof. Take a $c_b \in \mathbb{R}$ depending on the affine building such that for each $r \in \mathbb{N}$ the number of simplices in $C_r \cap \mathbb{A}$ is bounded by $c_b r^n$.

Let $h \in G$ be such that $\gamma \in Z_G(hSh^{-1})$. Combining Theorem 3.12 and Corollary 3.13 results in $\mathrm{tr}(\gamma, V) = \tau_{hC_{\mathrm{ht}(\Phi)\mathrm{sd}(\gamma)}}(\gamma)$. The number of simplices in $hC_{\mathrm{ht}(\Phi)\mathrm{sd}(\gamma)} \cap h\mathbb{A}$ is bounded by $c_b(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma) + 1)^n$. By Theorem 3.4 the number of γ-fixed simplices in $hC_{\mathrm{ht}(\Phi)\mathrm{sd}(\gamma)}$ is bounded by $c_b(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma) + 1)^n |D(\gamma)|^{-\frac{1}{2}}$. By Lemma 3.14 $|\mathrm{tr}(\rho(\gamma), V^{U_\sigma^e})| \leq C$ for all $\gamma \in gU_x^0$ and $\sigma \in \mathcal{B}^\gamma$. Thus

$$\begin{aligned} \mathrm{tr}(\gamma, V) = \tau_{hC_{\mathrm{ht}(\Phi)\mathrm{sd}(\gamma)}}(\gamma) &= \sum_{\sigma \in (hC_{\mathrm{ht}(\Phi)\mathrm{sd}(\gamma)})^\gamma} (-1)^{\dim \sigma} \epsilon_\sigma(\gamma) \mathrm{tr}(\rho(g), V^{U_\sigma^e}) \\ &\leq Cc_b(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma) + 1)^n |D(\gamma)|^{-\frac{1}{2}}. \end{aligned}$$

$\square$

Now we have an upper bound for the trace of the compact regular elements in $Z_G(S)$ in a neighborhood of a compact element of G. For a general regular element in $Z_G(S)$ in a neighborhood of a general element of G we use Casselman's method to compute the character.

Let P be an $\mathbb{F}$-parabolic subgroup of G, N its unipotent radical and M a Levi factor of P. For a representation (ρ, V) of G define

$$V(N) := \langle v - \rho(n)v : v \in V, n \in N \rangle$$

and $V_N := V/V(N)$. Now M acts on V_N via the action of M on V. The action of M on V_N is denoted by ρ_M. The M-module (ρ_M, V_N) is called the Jacquet module of V.

For $g \in G$ we have the *parabolic subgroup contracted by* g:

$$P_g := \{p \in G : \{g^n p g^{-n} : n \in \mathbb{N}\} \text{ is bounded}\} \text{ and}$$
$$M_g := \{p \in G : \{g^n p g^{-n} : n \in \mathbb{Z}\} \text{ is bounded}\}.$$

By [MS12, Proposition 2.3] P_g is a parabolic subgroup of G, M_g is a Levi subgroup and g, viewed as element of M_g, is compact. Roughly speaking, the center of M_g is larger than that of G in such a way that g is compact modulo this enlarged center.

Definition 3.16. *Let $g \in G$. We define the* displacement function $d_g : \mathcal{B}_e \to \mathbb{R}$ *by $d_g(x) := d(x, gx)$. Let $d(g) := \inf_{x \in \mathcal{B}_e} d_g(x)$.*

Let l be a line contained in $\mathbb{A}_e$. Let Φ_l be the set of roots α of S such that $\langle \alpha, \cdot \rangle$ is constant on l. Let M_l be the Levi subgroup of G generated by $Z_G(S)$ and the groups U_α for $\alpha \in \Phi_l$.

Lemma 3.17. *Let M be a Levi subgroup of a parabolic subgroup of G. Let S be a maximal split torus in M.*
The regular semisimple elements in ${}^G Z_G(S) \cap M$ are the regular semisimple elements in ${}^M Z_M(S) \cap M$.

Proof. If $\gamma \in {}^G Z_G(S) \cap M$ and γ is regular, then $Z_M^0(\gamma)$ is a maximal torus of M. Since the ranks of G and M are the same, $Z_M^0(\gamma)$ is also a maximal torus of G. Now $Z_G^0(\gamma)$ is a maximal torus of G, so $Z_G^0(\gamma) = Z_M^0(\gamma)$. Take $g \in G$ such that $\gamma \in Z_G(gSg^{-1})$. Thus $gSg^{-1} < Z_G^0(\gamma) = Z_M^0(\gamma)$. Since gSg^{-1} is a maximal split torus of G, it is also a maximal split torus of M. Since M is reductive, maximal split tori in M are conjugate over M. So there is $m \in M$ such that $gSg^{-1} = mSm^{-1}$. Thus $\gamma \in Z_M(mSm^{-1})$. □

The following is in the extended building.
For the moment let $g \in G$ be non-compact modulo the center. Thus $d(g) \neq 0$. Let l be a line in $\mathcal{B}_e$ on which g acts by translations. Such a line exists by [DeB02b, Lemma 3.4.4]. Let S be a maximal split-torus such that l is in the apartment of S. By [DeB02b, Lemma 3.4.4] $M_g = M_l$. So in particular $S \subset M_g$. Take $x \in l$, then $d_g(x) = d(g)$. Let $H = P_{[x,gx]}$. From the proof of [DeB02b, Lemma 3.4.7] we see that

$$d(gh) = d(g) \text{ for all } h \in H. \tag{3.2}$$

Lemma 3.18. *Let $h \in H$. The group M_{gh} is conjugate to M_g by an element of H.*
If, in addition, $gh \in M_g$, then $M_{gh} = M_g$.

Proof. Since $h \in H$, $d(gh) = d(x, gx) = d(x, ghx)$. By the proof of [DeB02b, Lemma 3.4.4] there is a line l' such that the points $(gh)^n x$ for $n \in \mathbb{Z}$ are on l'. This line lies in an apartment $\mathbb{A}'$. Now $[x, gx] \subset l \cap l'$. By [BT72, 7.4.9] there is $h_o \in H$ such that $h_o\mathbb{A} = \mathbb{A}'$. In the apartment $\mathbb{A}$ (respectively $\mathbb{A}'$) there is only one way to continue the line segment $[x, gx]$, namely, l (l' , respectively). Since h_o fixes $[x, gx]$ and maps lines to lines, we have $h_o l = l'$. So

$$M_{gh} = M_{l'} = M_{h_o l} = h_o M_l h_o^{-1} = h_o M_g h_o^{-1}.$$

Assume that $g' := gh \in M_g$. Since g' fixes x in $\mathcal{B}_a(M_g)$, g' is compact modulo the center of M_g. Since g' is compact modulo the center, if $m \in M_g$ then $\{g'^n m g'^{-n} : n \in \mathbb{Z}\}$ is bounded. So $M_g \subset M_{g'}$. Thus $g \in M_{g'}$. Since g fixes x in $\mathcal{B}_a(M_{g'})$, also $M_{g'} \subset M_g$. So $M_{gh} = M_{g'} = M_g$. □

Proposition 3.19. *For every $g \in G$ there exists a constant $C \in \mathbb{R}$, such that for all semisimple regular $\gamma \in {}^G Z_G(S) \cap gP_{[x,gx]}$:*

$$|\mathrm{tr}(\gamma, V)| \le C(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma) + 1)^{\dim \mathbb{A}_a} |D(\gamma)|^{-\frac{1}{2}}.$$

Proof. If g is compact modulo the center we can use Proposition 3.15.
Assume that g is not compact. Then $d(g) \neq 0$.
Let $H = P_{[x,gx]}$ for some $x \in \mathcal{B}_e$ such that $d_g(x) = d(g)$. By conjugating g we may assume that $g^z x \in \mathbb{A}_e$ for all $z \in \mathbb{Z}$.

Let N_g be the unipotent radical of P_g. Let (V_{N_g}, ρ_{N_g}) be the Jacquet module of ρ, a representation of M_g.

To indicate the difference between the objects defined for G and M_g, those corresponding with M_g are labelled by M_g, e.g., $\mathcal{B}_e$ is the building of G and $\mathcal{B}_e(M_g)$ is the building of M_g, D_{M_g} is the Harish-Chandra function D for M_g.

By [DeB02b, Lemma 3.4.4] $\mathbb{A}_e$ is an apartment of $\mathcal{B}_e(M_g)$ and the image of x in $\mathcal{B}_a(M_g)$ is a g-fixed point.

Let γ be a semisimple regular element in ${}^G Z_G(S)$.
Assume that $\gamma \in M_g$ and $\gamma \in gP_x$. By Lemma 3.17 $\gamma \in {}^{M_g} Z_{M_g}(S) \cap M_g$. Also $\gamma \in gP_x \cap M_g = gP_x(M_g)$. Let P be a parabolic subgroup containing M_g and let N be the unipotent radical of P. By Proposition 3.15 applied to (ρ_N, V_N), there is a $C \in \mathbb{R}$ such that for all such γ with $N = N_\gamma$,

$$|\mathrm{tr}(\gamma, V_{N_\gamma})| \le C(\mathrm{ht}(\Phi)\mathrm{sd}_{M_g}(\gamma) + 1)^n |D_{M_g}(\gamma)|^{-\frac{1}{2}}.$$

This C can and will be chosen independently of P and N, because there are only finitely many parabolic subgroups containing M_g.
By Casselman [Cas77], $\mathrm{tr}(\gamma, V) = \mathrm{tr}(\gamma, V_{N_g})$. Thus for all $\gamma \in M_g$ with $\gamma \in gP_x$,

$$|\mathrm{tr}(\gamma, V)| \le C(\mathrm{ht}(\Phi_{M_g})\mathrm{sd}_{M_g}(\gamma) + 1)^n |D_{M_g}(\gamma)|^{-\frac{1}{2}}. \tag{3.3}$$

Lemma 3.20. *There exists a $C' \in \mathbb{R}_{>0}$ such that for all semisimple regular elements $\gamma \in gP_x \cap M_g$ one has $\frac{|D(\gamma)|}{|D_{M_g}(\gamma)|} \le C'$.*

Proof. We are going to construct a continuous function on M_g, which on the semisimple regular elements γ takes the value $\frac{D(\gamma)}{D_{M_g}(\gamma)}$.
Pick a basis $b_1, \ldots, b_n$ of $\mathfrak{g}$ such that $b_1, \ldots, b_{\dim M_g}$ is a basis for $\mathfrak{m}_g$. Let $g' \in M_g$. Write the matrix $\mathrm{Ad}(g')$ with respect to this basis. Let $\varphi(g)$ be the determinant of the sub-matrix of $\mathrm{Ad}(g')$ in the lower right corner of dimension $\dim G - \dim M_g$. Then clearly $\varphi : M_g \to \mathbb{F}$ is continuous.

Let γ be a semisimple regular element in M_g. Notice that the definition of φ is independent of the choice of a basis with the property that the first $\dim M_g$ basis elements are in $\mathfrak{m}_g$. Since γ is semisimple regular, $T_\gamma := Z_G^0(\gamma)$ is a maximal torus and contained in M_g. Choose as basis for $\mathfrak{g}$, a basis for $\mathfrak{t}_\gamma$, and the eigenvectors $\mathfrak{u}_\alpha$, $\alpha \in R(M_g, T_\gamma)$, supplemented with the eigenvectors $\mathfrak{u}_\beta$ for all $\beta \in R(G, T_\gamma) - R(M_g, T_\gamma)$. So

$$\varphi(\gamma) = \prod_{\beta \in R(G,T_\gamma) - R(M_g,T_\gamma)} 1 - \beta(\gamma) = \frac{\prod_{\alpha \in R(G,T_\gamma)} 1 - \alpha(\gamma)}{\prod_{\alpha \in R(M_g,T_\gamma)} 1 - \alpha(\gamma)} = \frac{D(\gamma)}{D_{M_g}(\gamma)}.$$

Since φ is continuous and $gP_x \cap M_g$ is compact, there is a C' such that $\frac{|D(\gamma)|}{|D_{M_g}(\gamma)|} \leq C'$ for all semisimple regular elements $\gamma \in gP_x \cap M_g$. $\square$

Continuation of the proof of Proposition 3.19. Combining Lemma 3.20 with the estimate of the trace (3.3), $\mathrm{sd}_{M_g}(\gamma) \leq \mathrm{sd}(\gamma)$ and $\mathrm{ht}(\Phi_{M_g}) \leq \mathrm{ht}(\Phi)$ we get for all semisimple regular $\gamma \in gP_x \cap M_g$:

$$\begin{aligned} |\mathrm{tr}(\gamma, V)| &\leq C(\mathrm{ht}(\Phi_{M_g})\mathrm{sd}_{M_g}(\gamma) + 1)^n |D_{M_g}(\gamma)|^{-\frac{1}{2}} \\ &\leq C\sqrt{C'}(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma) + 1)^n |D(\gamma)|^{-\frac{1}{2}}. \end{aligned} \tag{3.4}$$

Assume that $\gamma \in {}^G Z_G(S) \cap gP_{[x,gx]}$. Lemma 3.18 gives a $h \in H$ such that $hM_\gamma h^{-1} = M_g$. Now $h\gamma h^{-1} \in M_g$ and $h\gamma h^{-1} \in gP_x$, because $h\gamma h^{-1}x = h\gamma x = hgx = gx$. Thus by (3.4):

$$\begin{aligned} |\mathrm{tr}(\gamma, V)| = |\mathrm{tr}(h\gamma h^{-1}, V)| &\leq C\sqrt{C'}(\mathrm{ht}(\Phi)\mathrm{sd}(h\gamma h^{-1}) + 1)^n |D(h\gamma h^{-1})|^{-\frac{1}{2}} \\ &= C\sqrt{C'}(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma) + 1)^n |D(\gamma)|^{-\frac{1}{2}}. \end{aligned}$$ $\square$

3.5 An estimate for the Weyl integration formula

Let $T := Z_G^0(\gamma)$ be the maximal torus containing γ. Let $n := \dim \mathbb{A}_a$. Recall, S is a maximal split torus.
In this section we want to give an estimate of the Weyl integration formula. To be precise, we will show that for every $f \in C_c^\infty(G)$ there exists a $C \in \mathbb{R}$ such that for all semisimple regular $\gamma \in Z_G(S)$ the following inequality holds:

$$\left| \int_{T \backslash G} f(g^{-1}\gamma g) \mathrm{d}g \right| \leq C(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) + 1)^n |D(\gamma)|^{-\frac{1}{2}}.$$

For $g \in G$ define

$$\mathcal{B}(g) := \{x \in \mathcal{B}_e(G) \mid d_g(x) = d(g)\}.$$

Let $g \in G$ and $x \in \mathcal{B}(g)$. We will first give an estimate in the case that $f := 1_{gP_{[x,gx]}}$. Let $\gamma \in Z_G(S) \cap gP_{x,gx}$ be a semisimple regular element. By equation (3.2) $d(\gamma) = d(g) = d(\gamma x, x)$, so $x \in \mathcal{B}(\gamma)$. For simplicity we estimate the integral of $1_{\gamma P_x}$ instead of $1_{gP_{[x,gx]}}$. Let $\phi_{M_\gamma} : \mathcal{B}_e(M_\gamma) \to \mathcal{B}_a(M_\gamma)$ be the canonical projection.
The relation between the integral and points in the building is due to the fact that if $1_{\gamma P_x}(g^{-1}\gamma g) = 1$, then $gx \in \mathcal{B}(\gamma) \subset \mathcal{B}_e(M_\gamma)$, since

$$d(gx, \gamma gx) = d(gx, g\gamma x) = d(x, \gamma x) = d(\gamma).$$

So we need to identify the elements in $Gx \cap \mathcal{B}(\gamma)$. Or more precisely, the T-orbits in $Gx \cap \mathcal{B}(\gamma)$, because we are integrating over $T\backslash G$. To give an upper bound for the number of T-orbits in $Gx \cap \mathcal{B}(\gamma)$, we look at $\mathcal{B}_{a,x}(\gamma) = \phi_{M_\gamma}(Gx \cap \mathcal{B}(\gamma))$. Now $\mathcal{B}_{a,x}(\gamma)$ consists of γ-fixed points. After some technicalities we get an upper bound for the number of T-orbits of γ-fixed points. This upper bound can certainly be improved, since it takes the measure on $T\backslash G$ into account.

Let F be the fundamental domain of T in $\mathbb{A}_e$ defined by

$$F := \{x \in \mathbb{A}_e \mid \forall t \in T\ [d(x, \mathsf{O}) \leq d(x, t\mathsf{O})]\}.$$

Definition 3.21. *Let $x \in \mathbb{A}_a$ and $z \in \mathcal{B}_a$, then z is called* above *x if x is a vertex and $d(x, z) \leq d(v, z)$ for all vertices $v \in \mathbb{A}_a$.*
Let $x + y \in \mathbb{A}_e$ be a vertex, with $x \in \mathbb{A}_a$ and $y \in Y$. Let $z \in \mathcal{B}_e$. Then z is called above *$x + y$ and $x + y$ is called* below *z if $\phi(z)$ is above $\phi(x)$ and $d(z, x + y) \leq d(z, x + y')$ for all $y' \in Y$.*

Lemma 3.22. *Let $x + y, z + y' \in \mathbb{A}_e$, with $x, z \in \mathbb{A}_a$ and $y, y' \in Y$. If $u(z + y')$ is above $x + y$ for $u \in U_x = U_{x+y}$, then $y' = y$.*

Proof. Let $y'' \in Y$. Since $u \in U_x = U_{x+y''}$,

$$d(u(z + y'), x + y'') = d(z + y', x + y'').$$

Now $d(z + y', x + y'') = (d(z, x)^2 + d(y', y'')^2)^{\frac{1}{2}}$. Therefore $y = y'$. □

Lemma 3.23. *Let G and H be unimodular groups such that H is a closed subgroup of G. Let K be an open compact subgroup of G. Suppose that the measures of G, $H\backslash G$ and H are invariant and chosen in such a way that $\mu_H(H \cap K) = \mu_{H\backslash G}(HK) = \mu_G(K) = 1$. Then, for any $g \in G$,*

$$\mu_{H\backslash G}(HgK) := \frac{[H \cap K : H \cap K \cap gKg^{-1}]}{[H \cap gKg^{-1} : H \cap K \cap gKg^{-1}]}.$$

Proof. See [Ren10, II.3.9] for a proof of the existence of a G-invariant measure on $H\backslash G$.

$$\int_H 1_{gK}(hg)\mathrm{d}h = \int_H 1_{gKg^{-1}}(h)\mathrm{d}h = \mu_H(gKg^{-1} \cap H).$$

Thus $\int_H 1_{gK}(hg')\mathrm{d}h = \mu_H(gKg^{-1} \cap H)1_{HgK}(g')$ for all $g' \in G$.
By the choice of the measure on H we have:

$$\mu_H(gKg^{-1} \cap H) = \frac{[gKg^{-1} \cap H : K \cap gKg^{-1} \cap H]}{[K \cap H : K \cap gKg^{-1} \cap H]}.$$

Since $\int_{H\backslash G}\int_H 1_{gK}(hx)\mathrm{d}h\mathrm{d}x = \int_G 1_{gK}(x)\mathrm{d}x = \mu_G(gK) = 1$,

$$\int_{H\backslash G} 1_{HgK}(x)\mathrm{d}x = \frac{1}{\mu_H(gKg^{-1}\cap H)}\int_{H\backslash G}\int_H 1_{gK}(hx)\mathrm{d}h\mathrm{d}x$$
$$= \frac{[K\cap H : K\cap gKg^{-1}\cap H]}{[gKg^{-1}\cap H : K\cap gKg^{-1}\cap H]}. \qquad \square$$

Let $K := P_x$. Take the measures on G, T and $T\backslash G$ as in Lemma 3.23.

$$L := \{y\in\mathcal{B}_e \mid y \text{ is above a vertex of } F\}.$$
$$L_\gamma := \{y\in L \mid y\in Gx \text{ and } y\in\phi_{M_\gamma}^{-1}(\mathcal{B}_a^\gamma)\}.$$

Lemma 3.24. $\int_{T\backslash G} 1_{\gamma P_x}(g^{-1}\gamma g)\mathrm{d}g \le |L_\gamma|$.

Proof. We will prove the following (in)equalities:

$$\int_{T\backslash G} 1_{\gamma P_x}(g^{-1}\gamma g)\mathrm{d}g = \sum_{g\in T\backslash G/P_{[x,\gamma x]}} 1_{P_x}(\gamma^{-1}g^{-1}\gamma g)\mu_{T\backslash G}(TgP_{[x,\gamma x]}) \tag{3.5}$$
$$\le \sum_{g\in T\backslash G/P_x}\mu_{T\backslash G}(TgP_x)1_x(gx) \tag{3.6}$$
$$\le \sum_{g\in(T\cap P_x)\backslash G/P_x \mid gx\in L_\gamma}\mu_{T\backslash G}(TgP_x) \tag{3.7}$$
$$\le \sum_{g\in(T\cap P_x)\backslash G/P_x \mid gx\in L_\gamma}|(T\cap P_x)gx| \tag{3.8}$$
$$= |L_\gamma|. \tag{3.9}$$

Since for $g'\in TgP_{[x,\gamma x]}$ we have

$$\gamma gx = g\gamma x \Leftrightarrow \gamma g'x = g'\gamma x,$$

the function $g\mapsto 1_{\gamma P_x}(g^{-1}\gamma g)$ is constant on double cosets $T\backslash G/P_{[x,\gamma x]}$. Therefore, we have equality (3.5).
Define $1_x : \mathcal{B}_e\to\mathbb{R}$ by

$$1_x(y) := \begin{cases} 1 & \exists g\in G[y = gx\wedge\gamma gx = g\gamma x], \\ 0 & \text{otherwise.}\end{cases}$$

Now $1_x(gx) = 1$ if and only if there exists an $h\in gP_x$ such that $1_{P_x}(\gamma^{-1}h^{-1}\gamma h) = 1$. Also $1_x(y) = 1_x(ty)$ for all $t\in T$. So

$$\sum_{h\in T\backslash TgP_x/P_{[x,\gamma x]}} 1_{P_x}(\gamma^{-1}h^{-1}\gamma h)\mu_{T\backslash G}(ThP_{[x,\gamma x]})$$
$$\le \sum_{h\in T\backslash TgP_x/P_{[x,\gamma x]}} 1_x(hx)\mu_{T\backslash G}(ThP_{[x,\gamma x]}) = 1_x(gx)\mu_{T\backslash G}(TgP_x).$$

This gives inequality (3.6).
For every coset Tg there exists a $g'\in Tg$ such that $g'x\in L$. If moreover $1_x(gx) = 1$,

then $g'x \in \mathcal{B}(\gamma)$. So $g'x \in L_\gamma$ and inequality (3.7) follows.
From Lemma 3.23 and $gP_xg^{-1} = P_{gx}$ we get inequality (3.8):

$$\begin{aligned}\mu_{T\backslash G}(TgP_x) &= \frac{[T\cap P_x : T\cap P_x \cap gP_xg^{-1}]}{[gP_xg^{-1}\cap T : T\cap P_x\cap gP_xg^{-1}]}\\ &\leq [T\cap P_x : T\cap P_x\cap gP_xg^{-1}] = |(T\cap P_x)gx|.\end{aligned}$$

The group $T \cap P_x$ fixes $\mathbb{A}_e$ pointwise and commutes with γ, so it acts on L_γ. So the sum in (3.8) is over the $(T\cap P_x)$-orbits in L_γ. Each orbit contributes to the sum the number of elements in that orbit. Thus the sum is the number of elements in L_γ. Therefore equality (3.9) holds. □

3.5.1 γ-Fixed points in the reduced building

In this subsection we assume that $\gamma \in Z_G(S)$ is a compact semisimple regular element of T.
Define $\Phi := \Phi(G,S)$ and $\tilde{\Phi} := \Phi(\mathcal{G},\mathcal{T})$. Let $\rho : \tilde{\Phi} \to \Phi \cup \{0\}$ be the canonical projection.
Define $n := \dim \mathbb{A}_a$.
The goal of this subsection is to prove the following theorem:

Theorem 3.25. *There is $c \in \mathbb{R}$ such that for all vertices $x \in \mathbb{A}_a$ and $\gamma \in T \cap P_{\mathsf{O}}$ the following holds: the number of vertices fixed by γ above x is bounded by $c(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) + 1)^n |D(\gamma)|^{-\frac{1}{2}}$.*

Let C be a Weyl chamber of $\mathbb{A}_a$ with vertex O, $\mathcal{C}$ the cone of C, $\Delta = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots associated to $\mathcal{C}$, and Φ^+ the set of positive roots.
Define for each simple root α_i a vertex a_i in $\mathbb{A}_a$ in the following way. Let $\Gamma \subset \Delta$ be the connected part of α_i in the Dynkin diagram. Let $\beta_0 := \sum_{\alpha_j\in\Gamma} c_j\alpha_j$ be the longest positive root in the root system generated by Γ. Define a_i to be the vertex in $\mathbb{A}_a$ such that $\alpha_j(a_i) = \frac{\delta_{ij}}{c_i}$.

Lemma 3.26. *For all $i \in \{1,\dots,n\}$ one has $d(\mathsf{O}, x + ta_i) > d(\mathsf{O}, x)$ for $t \in \mathbb{R}_{>0}$ and $x \in \mathcal{C}$.*

Proof. Recall that $d(\mathsf{O}, x) = \langle x, x\rangle = \sum_{\alpha\in\Phi^+} \alpha(x)^2$. Since $\alpha(x), \alpha_i(ta_i) > 0$ and $\alpha(ta_i) \geq 0$ for all $\alpha \in \Phi^+$, $\sum_{\alpha\in\Phi^+} \alpha(x + ta_i)^2 > \sum_{\alpha\in\Phi^+} \alpha(x)^2$. □

Lemma 3.27. *Let $x \in \mathbb{A}_a$ be a vertex. Assume that for $y = x + \sum_{j=1}^n n_jc_ja_j \in x + \mathcal{C}$ one has $n_i = \alpha_i(y - x) \geq \mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) + 1$ for some $i \in \{1,\dots,n\}$. Let $u \in U^- \cap U_x$. If uy is fixed by γ, then $d(uy, x + c_ia_i) < d(uy, x)$. So if uy is fixed by γ, then uy is not above x.*

Proof. Let $\beta \in \tilde{\Phi}$. Since uy is fixed by γ, $\nu(u_{-\beta}) \geq \beta(y) - \mathrm{ht}(\beta)\mathrm{sd}(\gamma)$ [MS12, Proposition 4.2]. Let $\alpha \in \Phi^+$. Write

$$u_{-\alpha} = \prod_{\beta\in\rho^{-1}(\alpha)} u_{-\beta} \prod_{\beta\in\rho^{-1}(2\alpha)} u_{-\beta}.$$

Now $\nu(u_\alpha) = \min\{\nu(u_{-\beta}) : \beta \in \rho^{-1}(\alpha)\} \cup \{\nu(u_{-\beta})/2 : \beta \in \rho^{-1}(2\alpha)\}$. The lower bound for $\nu(u_{-\beta})$ and $\beta(y) = \rho(\beta)(y)$ give that

$$\nu(u_{-\alpha}) \geq \alpha(y) - \mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma).$$

Let $\alpha \in \Phi^+$ with a non-zero coefficient for α_i in the decomposition of α as linear combination of the simple roots in Δ. So $\alpha = \sum_{j=1}^n d_j \alpha_j$ and $d_i \geq 1$.

$$\begin{aligned}\nu(u_{-\alpha}) &\geq \alpha(y) - \mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) = \alpha(x) + \alpha(y-x) - \mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) \\ &\geq \alpha(x) + d_i\alpha_i(y-x) - \mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) \\ &\geq \alpha(x) + d_i(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma)+1) - \mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) \geq \alpha(x) + d_i.\end{aligned}$$

For all $\alpha \in \Phi^+$ one has $\nu(u_{-\alpha}) \geq \alpha(x)$, since u fixes x. Therefore, with the previous inequality, $\nu(u_{-\alpha}) \geq \alpha(x + c_i a_i)$ for all $\alpha \in \Phi^+$. We conclude that u fixes $x + c_i a_i$. Hence

$$d(uy, x + c_i a_i) = d(y, u^{-1}(x + c_i a_i)) = d(y, x + c_i a_i) = d(y - c_i a_i, x).$$

Since $n_i \geq 1$, $y - c_i a_i \in x + \mathcal{C}$. So by Lemma 3.26

$$d(uy, x + c_i a_i) = d(y - c_i a_i, x) < d(y, x) = d(uy, x).$$

Since $\alpha_k(c_i a_i) = \frac{\delta_{ki} c_i}{c_i} \in \mathbb{N}$ for all simple roots α_k, the translation $y \mapsto y + c_i a_i$ is an automorphism of the apartment. So $x + c_i a_i$ is a vertex in $\mathbb{A}_a$. □

For $\alpha \in \Phi$ define n_α to be the smallest $r \in \mathbb{R}_{>0}$ such that $U_{\alpha,r} \neq U_{\alpha,r+}$. For $r \in \mathbb{R}$ define the α-*ceiling* as: $\lceil r \rceil_\alpha := \min\{z \in n_\alpha \mathbb{Z} \mid z \geq r\}$.

Lemma 3.28. *Let $x \in \mathbb{A}_a$ and $y \in \overline{C}$. There is a system of positive roots Φ^{++} such that:*
$-\alpha(x) \geq -\alpha(y)$ and $\lceil -\alpha(x) \rceil_\alpha \geq \lceil f_C(\alpha) \rceil_\alpha$ for all $\alpha \in \Phi^{++}$
$-\alpha(x) \leq -\alpha(y)$ and $\lceil -\alpha(x) \rceil_\alpha \leq \lceil f_C(\alpha) \rceil_\alpha$ for all $\alpha \in -\Phi^{++} = \Phi^{--}$

Proof. First we construct of $\Phi^{++} \subset \Phi$, then it will be proven that it is a system of positive roots that satisfies the requirements. For $\alpha \in \Phi^+$ the following rules decide whether $\alpha \in \Phi^{++}$ or $-\alpha \in \Phi^{++}$.

If $0 < \alpha(y) \leq n_\alpha$:

$$\begin{aligned}\alpha(x) < \alpha(y) &\Rightarrow +\alpha \in \Phi^{++}, \\ \alpha(x) \geq \alpha(y) &\Rightarrow -\alpha \in \Phi^{++}.\end{aligned}$$

If $0 = \alpha(y)$:

$$\begin{aligned}\alpha(x) \leq \alpha(y) &\Rightarrow +\alpha \in \Phi^{++}, \\ \alpha(x) > \alpha(y) &\Rightarrow -\alpha \in \Phi^{++}.\end{aligned}$$

By definition of C one has $\lceil f_C(\alpha) \rceil_\alpha = n_\alpha$ if $\alpha \in \Phi^-$ and 0 if $\alpha \in \Phi^+$.

First we check that the roots of Φ^{++} satisfy the requirements:
Certainly, if $\alpha \in \Phi^{++}$, then $-\alpha(x) \geq -\alpha(y)$.
Let $\alpha \in \Phi^+$.
If $\alpha \in \Phi^{++}$, one has $\alpha(x) < n_\alpha$.
If $\alpha(x) < n_\alpha$, then $\lceil -\alpha(x) \rceil_\alpha \geq 0 = \lceil f_C(\alpha) \rceil_\alpha$ and $\lceil \alpha(x) \rceil_\alpha \leq n_\alpha = \lceil f_C(-\alpha) \rceil_\alpha$.
If $-\alpha \in \Phi^{++}$, one has $\alpha(x) > 0$.
If $\alpha(x) > 0$, then $\lceil \alpha(x) \rceil_\alpha \geq n_\alpha = \lceil f_C(-\alpha) \rceil_\alpha$ and $\lceil -\alpha(x) \rceil_\alpha \leq 0 = \lceil f_C(\alpha) \rceil_\alpha$.

Thus for $\alpha \in \Phi^{++}$, one has $\lceil -\alpha(x) \rceil_\alpha \geq \lceil f_C(\alpha) \rceil_\alpha$ and for $\alpha \in \Phi^{--}$ one has $\lceil -\alpha(x) \rceil_\alpha \leq \lceil f_C(\alpha) \rceil_\alpha$.

By definition of Φ^{++}, if $-\alpha(x) > -\alpha(y)$, then $\alpha \in \Phi^{++}$.

Clearly, half of the roots are in Φ^{++} and $\Phi^{++} \cap -\Phi^{++} = \emptyset$. Therefore, it is now enough to show that if $\alpha, \beta \in \Phi^{++}$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi^{++}$.
Let $\alpha, \beta \in \Phi^+$. Let $i, j \in \{-, +\}$. Assume that one has $i\alpha + j\beta \in \Phi$ and $i\alpha, j\beta \in \Phi^{++}$. Case by case it can be shown that $i\alpha + j\beta \in \Phi^{++}$. $\square$

Theorem 3.29. *Let $y \in \mathbb{A}_a$.*
Define $\Pi := \{\Psi \subset \Phi \mid \Psi$ is a system of positive roots of $\Phi\}$. Define for $\Psi \in \Pi$ the group U^Ψ as the group generated by U_α for $\alpha \in \Psi$. Then

$$\mathcal{B}(G) = \bigcup_{\Phi^+ \in \Pi} \{ux : x \in \mathbb{A}_a, u \in U_y^{-\Phi^+} \mid \forall_{\alpha \in \Phi^+}\ \alpha(x) \geq \alpha(y)\}.$$

Proof. (See [MS12, §4.1]) Let $x \in \mathcal{B}(G)$ and choose a retraction ρ of $\mathcal{B}(G)$ to $\mathbb{A}_a$ centered in C. Take Φ^{++} a set of positive roots such that $-\alpha(\rho(x)) \leq -\alpha(y)$ and $\lceil f_{\rho(x)}(\alpha)) \rceil_\alpha \leq \lceil f_C(\alpha) \rceil_\alpha$ for $\alpha \in -\Phi^{++}$. Let D be a chamber in $\mathbb{A}_a$ whose closure contains $\rho(x)$ and for $\alpha \in \Phi^{--}$ one has $\lceil f_D(\alpha) \rceil_\alpha = \lceil \alpha(x) \rceil_\alpha$. Now $\lceil f_C(\alpha) \rceil_\alpha \geq \lceil f_{\rho(x)}(\alpha) \rceil_\alpha = \lceil f_D(\alpha) \rceil_\alpha$ for $\alpha \in \Phi^{--}$. Therefore $U_C^{--} \subset U_D^{--}$. Since $N_C = N_D$, one has $P_C \subset U_C^{++} P_D$. Because P_C acts transitively on the sets of apartments containing C there exists $u \in U_C^{++}$ such that $x = u\rho(x)$. $\square$

(Notice that with the same proof Theorem 3.29 holds with $\mathbb{A}_e$ substituted for $\mathbb{A}_a$.)

Now we have all the ingredients to prove Theorem 3.25.

Proof of Theorem 3.25. Let $x \in \mathbb{A}_a$ be a vertex and let z be a vertex above x fixed by γ. According to Theorem 3.29 there is a positive root system Φ^+ and $u \in U^-$ such that $z = uy$ with $y \in \mathbb{A}_a$ and $\alpha(y) \geq \alpha(x)$ for $\alpha \in \Phi^+$. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots of Φ^+. Define for each root α_i a vertex a_i in $\mathbb{A}_a$ in the following way. Let $\mathcal{C} := \{y \in \mathbb{A}_a : \alpha(y) > 0$ for all $\alpha \in \Phi^+\}$. Hence $y = x + \sum_{i=1}^n n_i c_i a_i$ with $n_i \in \mathbb{R}_{\geq 0}$. Since γ fixes uy and uy is above x, according to Lemma 3.27 $n_i < \mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) + 1$ for all $i \in \{1, \ldots, n\}$. Since $\dim \mathbb{A}_a = n$, there is $c \in \mathbb{R}$ such that for all $\gamma \in T \cap P_{\mathsf{O}}$ the number of vertices in $y \in \mathbb{A}_a \cap (x + \mathcal{C})$ with $\alpha_i(y - x) < \mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) + 1$ is bounded by $c(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) + 1)^n$.
By Theorem 3.4

$$\#\{uy : u \in U^- \cap P_x \mid \gamma uy = uy\} \leq |D(\gamma)|^{-\frac{1}{2}}.$$

Therefore, there is a $c \in \mathbb{R}$ such that for all $\gamma \in T \cap P_{\mathsf{O}}$ and all vertices $x \in \mathbb{A}_a$ the number of vertices fixed by γ and above x is bounded by $c(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma)+1)^n |D(\gamma)|^{-\frac{1}{2}}$. $\square$

Define the fundamental domain F_a for the action of S on $\mathbb{A}_a$ as follows:

$$F_a := \{x \in \mathbb{A}_a \mid \forall s \in S[d(x, \mathsf{O}) \leq d(x, s\mathsf{O})]\}.$$

For $\gamma \in Z_G(S)$ and $w \in \mathcal{B}_a$ let

$$L_{a,\gamma} := \{x \in Gw \mid x \text{ is above a vertex in } F_a \text{ and } \gamma x = x\}.$$

Corollary 3.30. *There is $c \in \mathbb{R}$ such that for all semisimple regular $\gamma \in Z_G(S) \cap P_w$:*

$$|L_{a,\gamma}| \leq c(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) + 1)^n |D(\gamma)|^{-\frac{1}{2}}.$$

Proof. Let $N \in \mathbb{N}$ be the number of vertices in F_a and C be the C of Theorem 3.25. Then $c := NC$ will do. □

3.5.2 An upper bound for the Weyl integral

Theorem 3.31. *Let $h \in G$ and $x \in \mathcal{B}(h)$. Then there is $C \in \mathbb{R}$ such that for all regular semisimple $\gamma \in {}^G Z_G(S)$*

$$\int_{Z_G^0(\gamma)\backslash G} 1_{hP_{[x,hx]}}(g^{-1}\gamma g)\mathrm{d}g \leq C(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) + 1)^n |D(\gamma)|^{-\frac{1}{2}}.$$

Proof. By conjugating h with a suitable element of G, x can be moved to $\mathbb{A}_e$. Both sides are invariant under conjugation with G. So without loss of generality we assume that $\gamma \in Z_G(S)$. Define $T := Z_G^0(\gamma)$. If the integral is non-zero, there is $g \in G$ such that $g^{-1}\gamma g \in hP_{[x,hx]}$. Then $d(\gamma) = d(g^{-1}\gamma g) = d(h)$ by Equation (3.2). Thus without loss of generality we assume that $d(\gamma) = d(h)$.

Since $\gamma \in Z_G(S)$ and $x \in \mathbb{A}_e(S)$, $x \in \mathcal{B}(\gamma)$. Thus by Lemma 3.24

$$\int_{T\backslash G} 1_{\gamma P_x}(g^{-1}\gamma g)\mathrm{d}g \leq |L_\gamma|.$$

So it is enough to show that $|L_\gamma| \leq C(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) + 1)^n |D(\gamma)|^{-\frac{1}{2}}$.

Let M be a Levi subgroup, such that $Z_G(S) \subset M$. Define

$$Z_G(S)_M := \{\gamma \in Z_G(S) \mid d(\gamma) = d(h), \gamma \text{ is regular semisimple and } M_\gamma = M\}.$$

We will give an upper bound for $|L_\gamma|$ for all $\gamma \in Z_G(S)_M$.

Lemma 3.32. *Let $x \in \mathcal{B}_e(G)$ and let M a Levi subgroup. Then $Gx \cap \mathcal{B}_e(M)$ consists of finitely many M-orbits.*

Proof. If the lemma holds for M it also holds for gMg^{-1}. Thus without loss of generality we assume that $S \subset M$. If $gx \in \mathcal{B}_e(M)$, there is $m \in M$ such that $mgx \in \mathbb{A}_e$. Thus every M-orbit may and will be represented by a point in $\mathbb{A}_e$. Let F_a be the fundamental domain of S in $\mathbb{A}_e$. Then every M-orbit has at least one point in F_a. Since F_a is bounded and there is $r \in \mathbb{R}$ such that $d(z, z') \geq r$ for distinct $z, z' \in Gx$, there are only finitely many points of Gx in F_a. So the number of M-orbits in $G \cap \mathcal{B}_e(M)$ is finite. □

Recall the canonical map $\phi_M : \mathcal{B}_e(M) \to \mathcal{B}_a(M)$. Define

$$L_{x,\gamma}(M) := \{y \in \phi_M(Gx \cap \mathcal{B}_e(M)) \mid y \text{ is above a vertex of } F_a \text{ and } \gamma y = y\}.$$

By Corollary 3.30 and Lemma 3.32 there is $c \in \mathbb{R}$ such that for all $\gamma \in Z_G(S)_M$:

$$|L_{x,\gamma}(M)| \leq c(\mathrm{ht}(\tilde{\Phi}_M)\mathrm{sd}_M(\gamma) + 1)^n |D_M(\gamma)|^{-\frac{1}{2}}.$$

Let $Y_M := \mathcal{B}_e(Z(M)) = \mathbb{A}_e(Z(M))$.
Then $\mathbb{A}_e(M) = \mathbb{A}_a(M) \oplus Y_M$.
Define $\pi_M : \mathcal{B}_e(M) \to Y_M$ by $(g, x+y) \mapsto (g, y)$, for $x \in \mathbb{A}_a(M)$ and $y \in Y_M$. Define $D := \pi_M(F_a)$.

Lemma 3.33. *There is c_0 only depending on M such that $|L_\gamma| \le c_0 |L_{x,\gamma}(M)|$.*

Proof. For $z \in L_{x,\gamma}(M)$ define

$$F(z) := \phi_M^{-1}(z) \cap L_\gamma.$$

Let $z' \in \mathbb{A}_a(M)$, $a \in \mathbb{A}_a(M)$ and $u \in U_a$, such that $z = uz'$ and z' is above a. Let $v \in \phi_M^{-1}(z) \cap L_\gamma$. Then there is $y \in Y_M$ such that $v = u(z'+y)$. Let $a + y' \in F_a$ such that $u(z+y)$ is above $a+y'$. By Lemma 3.22, then $y = y'$. Thus if $v \in \phi_M^{-1}(z) \cap L_\gamma$, then $u^{-1}v \in (z'+D) \cap Gx$.
Because there exists $r \in \mathbb{R}_{>0}$ such that $d(z, z') > r$ for all distinct $z, z' \in Gx$, there exists $N \in \mathbb{N}$ such that $|(z'+D) \cap Gx| \le N$ for all $z' \in \mathbb{A}_a(M)$.
Thus $|F(z)| \le N$ and the lemma follows. □

Continuation of the proof of Theorem 3.31. By Lemma 3.33 and Corollary 3.30 for all Levi subgroups M containing S, there is $C_M \in \mathbb{R}_{>0}$ such that for all $\gamma \in Z_G(S)$ with $M_\gamma = M$:

$$|L_\gamma| \le C_M(\mathrm{ht}(\tilde{\Phi}_{M_\gamma})\mathrm{sd}_{M_\gamma}(\gamma)+1)^n |D_{M_\gamma}(\gamma)|^{-\frac{1}{2}}.$$

By Lemma 3.20 and the fact that there are only finitely many Levi subgroups containing S there is $C \in \mathbb{R}$ such that for all $\gamma \in Z_G(S)$ with $d(\gamma) = d(h)$:

$$|L_\gamma| \le C(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma)+1)^n |D(\gamma)|^{-\frac{1}{2}}.$$ □

Proposition 3.34. *Let $f \in C_c^\infty(G)$ and let $\omega \subset G$ be a compact subset of G. Then there exists $C \in \mathbb{R}$ such that for all $\gamma \in Z_G(S) \cap \omega$*

$$\left| \int_{Z_G(\gamma)\backslash G} f(g^{-1}\gamma g)\mathrm{d}g \right| \le C(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma)+1)^{\dim \mathbb{A}_a} |D(\gamma)|^{-\frac{1}{2}}.$$

Proof. Let $\Omega \subset G$ be a compact subset. Let $T := Z_G^0(\gamma)$. Then there are $g_1, \dots, g_m \in G$ and $x_1, \dots, x_m \in \mathcal{B}_e$ such that $d_{g_i}(x_i) = d(g_i)$ and $\Omega \subset \bigcup_{i=1}^m g_i P_{[x_i, g_i x_i]}$. Therefore

$$\int_{T\backslash G} 1_\Omega(g^{-1}\gamma g)\mathrm{d}g \le \sum_{i=1}^m \int_{T\backslash G} 1_{g_i P_{[x_i, g_i x_i]}}(g^{-1}\gamma g)\mathrm{d}g.$$

So it is enough to give an estimate for $\int_{T\backslash G} 1_{g_i P_{[x_i, g_i x_i]}}(g^{-1}\gamma g)\mathrm{d}g$.
Take $h_i \in G$ such that $x_i \in h_i\mathbb{A}_e(S)$. Now apply Theorem 3.31 to $x \in \mathbb{A}_e(hSh^{-1})$ and $\gamma \in Z_G(S) \subset {}^G Z_G(hSh^{-1})$.
Since $C_c^\infty(G)$ is spanned as a $\mathbb{C}$-vector space by the 1_Ω with Ω a compact subset of G, the proposition follows. □

3.6 Local summability of the character on ${}^{G}T$ $(S \subset T)$

In this section we combine the upper bounds for the Weyl integration formula and for the character of the representation to show that the character is locally summable on ${}^{G}T$ for a maximal torus T containing a maximal split torus S. It turns out that it is enough to show that sd^k is locally summable on T. Inspired by Harish-Chandra [HC70, Lemma 43] we show that even $\mathrm{sd}^k|D|^{-\epsilon}$ is locally summable on every maximal $\mathbb{F}$-torus T of G for some $\epsilon > 0$ depending on T.

3.6.1 Local summability of $\mathrm{sd}^k|D|^{-\epsilon}$ on T

In the first part of this subsection, T is an arbitrary $\mathbb{F}$-torus (not necessarily contained in G).
Integrating a function in a small neighborhood of the identity in an $\mathbb{F}$-split torus can be translated to integrating a function in a small neighborhood of 0 in an $\mathbb{F}$-vector space. For a 1-dimensional $\mathbb{F}$-split torus, just apply the map $e : \mathcal{O} \to \mathcal{O}^\times$, $e(a) := 1 + \varpi a$. If $\chi \in X^*(T)$, then integrating the function $|\chi(t) - 1|^{-\epsilon}$ in a small neighborhood of id becomes integrating $|(1 + \varpi x)^n - 1|^{-\epsilon}$ over a small neighborhood of 0. To study the integral $|(1 + \varpi x)^n - 1|^{-\epsilon}$ over $\mathcal{O}$, we want to have an estimate for the measure of

$$\mathcal{O}_r := \{x \in \mathcal{O} \mid \nu((1 + \varpi x)^n - 1) \geq r\}$$

in $\mathcal{O}$. With this in mind, we study first

$$\mathcal{O}_r(f) := \{x \in \mathcal{O} \mid \nu(f(x)) \geq r\}$$

for a polynomial $f \in \mathcal{O}[x]$, with $f \neq 0$.
In the case that T is not an $\mathbb{F}$-split torus there is in general no polynomial bijection between $\mathcal{O}^m$ and a neighborhood of the identity. However, we are able to construct a surjective map from $\mathcal{O}_{\mathbb{E}}^n$ to Υ for some Galois extension $\mathbb{E}$ and compact subgroup Υ of T, using a generalized norm map $N_{\mathbb{E}/\mathbb{F}} : T(\mathbb{E}) \to T(\mathbb{F})$. This gives rise to the study of the measure of

$$\mathcal{O}_r^n(f) := \{x \in \mathcal{O}^n \mid \nu(f(x)) \geq r\}$$

in $\mathcal{O}^n$ for a polynomial $f \in \mathcal{O}_{\mathbb{E}}[x_1, \ldots, x_n]$, with $f \neq 0$.

For $f \in \mathbb{F}[x_1, \ldots, x_n]$, write $f = \sum_{a \in \mathbb{N}^n} c(a) \prod_{i=1}^n x_i^{a_i}$.
Define $m_i := \max\{l \in \mathbb{N} \mid \exists a \in \mathbb{N}^n [a_i = l \text{ and } c(a) \neq 0]\}$.
Define $m_f := \max_i m_i$.
Thus m_f is the highest number that occurs as a power of any x_i in the expression of f.
Thus for $f(x_1, x_2) := x_1 x_2^3 + x_1 x_2 + 2$ we have $m_1 = 1$, $m_2 = 3$ and $m_f = 3$.

Lemma 3.35. *Let $\mathbb{E}/\mathbb{F}$ be a finite field extension.*
Let $f \in \mathcal{O}_{\mathbb{E}}[x_1, \ldots, x_n]$ and $f \neq 0$. There exists $C \in \mathbb{R}_{>0}$ such that for all $r \in \mathbb{Q}$ and $N \in \mathbb{N}$ with $N \geq r$:

$$\frac{1}{q^{nN}} |\{x \in (\mathcal{O}_{\mathbb{F}}/\varpi^N \mathcal{O}_{\mathbb{F}})^n \mid \nu(f(x)) \geq r\}| \leq C N^{n-1} q^{-\frac{r}{m_f}}.$$

Proof. Since $f \in \mathcal{O}_{\mathbb{E}}[x_1, \ldots, x_n]$, to ask for $x \in (\mathcal{O}/\varpi^N\mathcal{O})^n$, whether $\nu(f(x)) \geq r$ makes sense if $N \geq r$.
We prove this lemma by induction on n.
Assume that $n = 1$, so $f(x) := \sum_{i=1}^m a_i x^i$, with $a_m \neq 0$. Take $\alpha_1, \ldots, \alpha_n$ in an algebraic closure of $\mathbb{E}$ such that $f(x) = a_m \prod_{i=1}^m (x - \alpha_i)$.
Assume that $\nu(f(x)) \geq r$. Then for some i, $\nu(a_m) + m\nu(x - \alpha_i) \geq r$. Thus $\nu(x - \alpha_i) \geq \frac{r - \nu(a_m)}{m}$.
So the number of $x \in \mathcal{O}_{\mathbb{F}}/\varpi^N\mathcal{O}_{\mathbb{F}}$ such that $\nu(f(x)) \geq r$ is bounded by $mq^{N - \frac{r-\nu(a_m)}{m}}$. Hence

$$\frac{1}{q^N}|\{x \in \mathcal{O}_F/\varpi^N\mathcal{O}_F \mid \nu(f(x)) \geq r\}| \leq mq^{-\frac{r-\nu(a_m)}{m}}.$$

Assuming that we know the Lemma for n, we will prove the Lemma for $n + 1$.
Let $m := m_f$. Without loss of generality assume that $m = m_{n+1}$. Take $g_0, \ldots, g_m \in \mathcal{O}_{\mathbb{E}}[x_1, \ldots, x_n]$ such that $f = \sum_{i=0}^m g_i x_{n+1}^i$. Then $g_m \neq 0$ and $m \geq m_{g_m}$. Now we apply the induction hypothesis on g_m. Take a $C \in \mathbb{R}_{>0}$ such that for all $r \in \mathbb{Q}$ and $N \in \mathbb{N}$ with $N \geq r$,

$$\frac{1}{q^{nN}}|\{x \in (\mathcal{O}_{\mathbb{F}}/\varpi^N\mathcal{O}_{\mathbb{F}})^n \mid \nu(g_m(x)) \geq r\}| \leq CN^{n-1}q^{-\frac{r}{m_{g_m}}}.$$

Define the following sets

$$\begin{aligned} V_r &:= \{x \in (\mathcal{O}/\varpi^N\mathcal{O})^n \mid \nu(g_m(x)) = r\}, \\ O_{r,s} &:= \{x \in (\mathcal{O}/\varpi^N\mathcal{O})^{n+1} \mid \nu(g_m(x_1, \ldots, x_n)) = s \text{ and } \nu(f(x)) \geq r\}. \end{aligned}$$

Define, for $x_1, \ldots, x_n \in \mathcal{O}/\varpi^N\mathcal{O}$ and $r \in \mathbb{Q}$, the set:

$$U_{x_1,\ldots,x_n,r} := \{x \in \mathcal{O}/\varpi^N\mathcal{O} \mid \nu(f(x_1, \ldots, x_n, x)) \geq r\}.$$

So

$$O_{r,s} = \{x \in (\mathcal{O}/\varpi^N\mathcal{O})^{n+1} \mid (x_1, \ldots, x_n) \in V_s \text{ and } x_{n+1} \in U_{x_1,\ldots,x_n,r}\}.$$

By the proof of the lemma in the case $n = 1$ we have

$$|U_{x_1,\ldots,x_n,r}| \leq mq^{N - \frac{r - \nu(g_m(x_1,\ldots,x_n))}{m}}$$

whenever $\nu(g_m(x_1, \ldots, x_n)) < N$.

Let $x_1, \ldots, x_n \in \mathcal{O}/\varpi^N\mathcal{O}$, such that $\nu(g_m(x_1, \ldots, x_n)) = s < N$. Then

$$\frac{1}{q^N}|U_{x_1,\ldots,x_n,r}| \leq mq^{-\frac{r-s}{m}}.$$

By the induction hypothesis on g_m we have

$$\frac{1}{q^{nN}}|V_s| \leq CN^{n-1}q^{-\frac{s}{m_{g_m}}}.$$

Thus

$$\begin{aligned} \frac{1}{q^{(n+1)N}}|O_{r,s}| &= \frac{1}{q^{(n+1)N}} \sum_{x \in V_s} |U_{x,r}| \leq \frac{1}{q^{nN}} \sum_{x \in V_s} mq^{-\frac{r-s}{m}} \\ &= \frac{1}{q^{nN}}|V_s|mq^{-\frac{r-s}{m}} \leq CN^{n-1}q^{-\frac{s}{m_{g_m}}}mq^{-\frac{r-s}{m}} \\ &\leq mCN^{n-1}q^{-\frac{r}{m}}. \end{aligned}$$

Let e be the ramification index of $\mathbb{E}/\mathbb{F}$. So

$$\begin{aligned}
&\frac{1}{q^{(n+1)N}}|\{x \in (\mathcal{O}_{\mathbb{F}}/\varpi^N\mathcal{O}_{\mathbb{F}})^{n+1} \mid \nu(f(x)) \geq r\}| \\
&\leq \frac{1}{q^{(n+1)N}}|\{x \in (\mathcal{O}_{\mathbb{F}}/\varpi^N\mathcal{O}_{\mathbb{F}})^{n+1} \mid \nu(g_m(x_1,\dots,x_n)) \geq N\}| + \sum_{i=0}^{eN-1} \frac{1}{q^{(n+1)N}}|O_{r,\frac{i}{e}}| \\
&\leq \frac{1}{q^{nN}}|\{x \in (\mathcal{O}_{\mathbb{F}}/\varpi^N\mathcal{O}_{\mathbb{F}})^{n} \mid \nu(g_m(x)) \geq N\}| + \sum_{i=0}^{eN-1} \frac{1}{q^{(n+1)N}}|O_{r,\frac{i}{e}}| \\
&\leq CN^{n-1}q^{-\frac{N}{m_{g_m}}} + \sum_{i=0}^{eN-1} mCN^{n-1}q^{-\frac{r}{m}} \\
&\leq CN^{n-1}q^{-\frac{r}{m}} + eNmCN^{n-1}q^{-\frac{r}{m}} \\
&\leq 2emCN^nq^{-\frac{r}{m}}.
\end{aligned}$$

□

Let $\mathbb{E}/\mathbb{F}$ be a finite Galois extension such that T is $\mathbb{E}$-split. Define the function $N_{\mathbb{E}/\mathbb{F}} : T(\mathbb{E}) \to T(\mathbb{F})$ as follows:

$$N_{\mathbb{E}/\mathbb{F}}(t) := \prod_{\sigma \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})} \sigma(t).$$

Since T is Abelian, $N_{\mathbb{E}/\mathbb{F}}(t)$ is invariant under the Galois action. Hence the image of $N_{\mathbb{E}/\mathbb{F}}$ lies in $T(\mathbb{F})$. The group $\mathrm{Gal}(\mathbb{E}:\mathbb{F})$ acts on $X^*(T)$ by

$$(\sigma \cdot \chi)(t) := \sigma(\chi(\sigma^{-1}(t))).$$

Let $m = \dim T$ and $n = [\mathbb{E}:\mathbb{F}]$.

Let $\chi_1, \dots, \chi_m$ be a basis for $X^*(T)$ and $X_1, \dots, X_m$ the dual basis for $X_*(T)$. Parametrize $T(\mathbb{E})$ by $(\mathbb{E}^\times)^n \to T(\mathbb{E})$:

$$a \mapsto \prod_{i=1}^m X_i(a).$$

Define $K := \{\prod_{i=1}^n X_i(a_i) : a_i \in 1 + \varpi\mathcal{O}_{\mathbb{E}}\}$.

Take $\alpha \in \mathbb{E}$ such that $\mathcal{O}_{\mathbb{E}} = \mathcal{O}_{\mathbb{F}}[\alpha]$.
Define $\alpha_i := \alpha^{i-1}$ for $i = 1, \dots, n$. So

$$\sum_{i=1}^n a_i\alpha_i \in \varpi^k\mathcal{O}_{\mathbb{E}} \Leftrightarrow \forall i[a_i \in \varpi^k\mathcal{O}_{\mathbb{F}}].$$

Write $\mathbb{E}$ as an $\mathbb{F}$-vector space with basis $1, \alpha_2, \dots, \alpha_n$. For $a \in \mathbb{E}^m$, we define, for $1 \leq i \leq m$ and $1 \leq j \leq n$, the elements $a_{ij} \in \mathbb{F}$ to be the coordinates of a_i with respect to this basis. Thus

$$a_i = \sum_{j=1}^n a_{ij}\alpha_i.$$

Define $\mathsf{p} : (\mathbb{F}^n - 0)^m \to T(\mathbb{E})$ by:

$$\mathsf{p}(a) := \prod_{i=1}^{m} X_i \left(\sum_{j=1}^{n} a_{ij}\alpha_j \right).$$

Lemma 3.36. *Let $\chi \in X^*(T)$. There exist $f, g \in \mathcal{O}_{\mathbb{E}}[x_{11}, \ldots, x_{mn}]$, such that*

$$\chi \circ N_{\mathbb{E}/\mathbb{F}} \circ \mathsf{p}(a) = \frac{f(a)}{g(a)}.$$

Moreover if $\mathsf{p}(a) \in K$, then $f(a), g(a) \in \mathcal{O}_{\mathbb{E}}^{\times}$.

Proof. Let $a \in \mathbb{E}^{\times}$, then

$$\chi \circ N_{\mathbb{E}/\mathbb{F}} \circ X_i(a) = \prod_{\sigma \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})} \chi(\sigma(X_i(a))) = \prod_{\sigma \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})} \sigma(a)^{z_\sigma},$$

where $z_\sigma = \langle \sigma^{-1} \cdot \chi, X_i \rangle$. An automorphism $\sigma \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})$ is, with $\mathbb{E}$ viewed as $\mathbb{F}$-vector space with basis $1, \alpha_2, \ldots, \alpha_n$, a polynomial map:

$$g_\sigma(x_1, \ldots, x_n) := \sum_{i=1}^{n} \sigma(\alpha_i) x_i.$$

Then for $a = \sum_{i=1}^{n} a_i \alpha_i$ with $a_i \in \mathbb{F}$ we have

$$g_\sigma(a_1, \ldots, a_n) = \sigma(a).$$

Since $\alpha \in \mathcal{O}_{\mathbb{E}}$, also $\sigma(\alpha^i) \in \mathcal{O}_{\mathbb{E}}$ for $i \geq 0$. Therefore $g_\sigma \in \mathcal{O}_{\mathbb{E}}[x_1, \ldots, x_n]$.
Thus

$$\chi \circ N_{\mathbb{E}/\mathbb{F}} \circ X_i \left(\sum_{j=1}^{n} a_j \alpha_j \right) = \prod_{\sigma \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})} g_\sigma(a_1, \ldots, a_n)^{z_\sigma} = \frac{f_i(a_1, \ldots, a_n)}{g_i(a_1, \ldots, a_n)},$$

where $f_i, g_i \in \mathcal{O}_{\mathbb{E}}[x_1, \ldots, x_n]$. The first part of the lemma follows.
If $a \in 1 + \varpi\mathcal{O}_{\mathbb{E}}$, then $\sigma(a) \in 1 + \varpi\mathcal{O}_{\mathbb{E}}$ for all $\sigma \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. Thus if $\mathsf{p}(a) \in K$, then $f(a), g(a) \in \mathcal{O}_{\mathbb{E}}^{\times}$. $\square$

Proposition 3.37. *Let $\chi \in X^*(T)$. There exists an $\epsilon > 0$ such that $|\chi(t) - 1|^{-\epsilon}$ is locally summable on $T(\mathbb{F})$.*

Before we prove this proposition, we first prove two lemmas. For $r \in \frac{1}{e}\mathbb{N}$ define

$$K_r := \{k \in K : \nu(\chi \circ N_{\mathbb{E}/\mathbb{F}}(k) - 1) \geq r\}.$$

Then K_r is a compact open subgroup of K.

Lemma 3.38. *There exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that*

$$\frac{1}{[K : K_r]} \leq c_1 \lceil r \rceil^{nm-1} q^{-\frac{r}{c_2}},$$

for all $r \in \frac{1}{e}\mathbb{N}$.

Proof. Take $f, g \in \mathcal{O}_{\mathbb{E}}[x_{11}, \dots, x_{mn}]$ as in Lemma 3.36.
Since the elements of $T(\mathbb{F})$ are invariant under the Galois action:

$$\chi \circ N_{\mathbb{E}/\mathbb{F}}|_{T(\mathbb{F})} = n\chi|_{T(\mathbb{F})}.$$

Since $T(\mathbb{F})$ is Zariski dense in $\mathcal{T}$, there is $t \in T(\mathbb{F})$ with $n\chi(t) \neq 1$. So there is $t \in T(\mathbb{F})$ with $\chi \circ N_{\mathbb{E}/\mathbb{F}}(t) \neq 1$. Thus $\frac{f(a)}{g(a)} \neq 1$.

Define $\mathbf{e} : \mathcal{O}_{\mathbb{E}} \to 1 + \varpi\mathcal{O}_{\mathbb{E}}$ by $\mathbf{e}(a) := 1 + \varpi a$.
Let $\mathsf{p}' : (\mathcal{O}_{\mathbb{F}}^n)^m \to K$ be defined by

$$\mathsf{p}'(a) := \prod_{i=1}^{m} X_i\left(\mathbf{e}\left(\sum_{j=1}^{n} a_{ij}\alpha_i\right)\right).$$

Then p' is a bijection. Now

$$\chi \circ N_{\mathbb{E}/\mathbb{F}} \circ \mathsf{p}'(a) = \frac{\psi(f)(a)}{\psi(g)(a)},$$

where $\psi : \mathbb{E}[x_{11}, \dots, x_{mn}] \to \mathbb{E}[x_{11}, \dots, x_{mn}]$ is the automorphism defined by

$$\psi(x_{ij}) := \begin{cases} 1 + \varpi x_{ij} & \text{if } j = 1, \\ \varpi x_{ij} & \text{otherwise.} \end{cases}$$

The bijection p' gives a set corresponding to K_r in $(\mathcal{O}_{\mathbb{F}}^n)^m$:

$$(\mathcal{O}_{\mathbb{F}}^n)_r^m := \mathsf{p}'^{-1}(K_r) = \left\{a \in (\mathcal{O}_F^n)^m \mid \nu\left(\frac{\psi(f)(a)}{\psi(g)(a)} - 1\right) \geq r\right\}.$$

Since $\psi(g)(x) \in \mathcal{O}^\times$ for all $x \in (\mathcal{O}_{\mathbb{F}}^n)^m$, we have

$$\nu\left(\frac{\psi(f)(x)}{\psi(g)(x)} - 1\right) = \nu(\psi(f)(x) - \psi(g)(x)).$$

Define $h(x) := \psi(f)(x) - \psi(g)(x)$, then $h \in \mathcal{O}_{\mathbb{E}}[x_{11}, \dots, x_{mn}]$ and

$$(\mathcal{O}_{\mathbb{F}}^n)_r^m = \{a \in ((\mathcal{O}_{\mathbb{F}})^n)^m \mid \nu(h(a)) \geq r\}.$$

Define $K^{(N)} := \{\prod_{i=1}^n X_i(a_i) : a_i \in 1 + \varpi^N\mathcal{O}_{\mathbb{E}}\}$.
Now $\mathsf{p}'(a)K^{(N)} = \mathsf{p}'(a')K^{(N)}$ if and only if $a_{ij} \equiv a'_{ij} \mod \varpi^{N-1}\mathcal{O}_{\mathbb{F}}$. Let $N \geq r$, then $K^{(N)} < K_r$. Thus

$$\frac{1}{[K : K_r]} = \frac{1}{q^{nm(N-1)}}|\{x \in (\mathcal{O}_{\mathbb{F}}/\varpi^{N-1}\mathcal{O}_{\mathbb{F}})^{mn} \mid \nu(h(x)) \geq r\}|.$$

By Lemma 3.35 there exists a C such that for all r and N with $N \geq r$,

$$\frac{1}{q^{nmN}}|\{x \in (\mathcal{O}_{\mathbb{F}}/\varpi^N\mathcal{O}_{\mathbb{F}})^{mn} \mid \nu(h(x)) \geq r\}| \leq CN^{nm-1}q^{-\frac{r}{m_h}}.$$

Take $N = \lceil r \rceil + 1$. Thus $\frac{1}{[K:K_r]} \leq C(\lceil r \rceil + 1)^{nm-1}q^{-\frac{r}{m_h}}$. $\square$

Define $\Upsilon := N_{\mathbb{E}/\mathbb{F}}(K)$. Since K is compact, Υ is a closed subgroup of $T(\mathbb{F})$. We have $\Upsilon < K$. Let $\Upsilon_r := \{s \in \Upsilon \mid \nu(\chi(s) - 1) \geq r\}$. Define

$$T(\mathbb{F})_r := \{t \in T(\mathbb{F}) : \nu(\chi(t) - 1) \geq r\}.$$

Lemma 3.39. $[K : K_r] = [\Upsilon : \Upsilon_r] \leq [T(\mathbb{F}) \cap K : T(\mathbb{F})_r \cap K]$.

Proof. Since $N_{\mathbb{E}/\mathbb{F}} : K \to \Upsilon$ is surjective,

$$[\Upsilon : \Upsilon_r] = [K : K_r \ker N_{\mathbb{E}/\mathbb{F}}].$$

If $k \in \ker N_{\mathbb{E}/\mathbb{F}}$, then $\chi \circ N_{\mathbb{E}/\mathbb{F}}(k) = \chi(1) = 1$. Thus $\ker N_{\mathbb{E}/\mathbb{F}} < K_r$. So $K_r \ker N_{\mathbb{E}/\mathbb{F}} = K_r$. Thus $[K : K_r] = [\Upsilon : \Upsilon_r]$.
Since $\Upsilon < T(\mathbb{F}) \cap K$ and $\Upsilon_r = T(\mathbb{F})_r \cap K \cap \Upsilon$,

$$[\Upsilon : \Upsilon_r] \leq [T(\mathbb{F}) \cap K : T(\mathbb{F})_r \cap K]. \qquad \square$$

Proof of Proposition 3.37. Let $t_0 \in T$. If $\chi(t_0) \neq 1$, then $|\chi(t) - 1|^{-\epsilon}$ is constant on a neighborhood of t_0. Thus in particular $|\chi(t) - 1|^{-\epsilon}$ is locally summable around t_0.
Assume that $\chi(t_0) = 1$ and $\int_K |\chi(t) - 1|^{-\epsilon} \mathrm{d}t < \infty$ for some open compact subgroup $K < T$. Since $\chi(t_0 t) = \chi(t)$ for $t \in K$,

$$\int_{t_o K} |\chi(t) - 1|^{-\epsilon} \mathrm{d}t = \int_K |\chi(t) - 1|^{-\epsilon} \mathrm{d}t < \infty.$$

So then $|\chi(t) - 1|^{-\epsilon}$ is locally summable around t_0. Thus it is enough to show that for some open compact subgroup K

$$\int_{K \cap T(\mathbb{F})} |\chi(t) - 1|^{-\epsilon} \mathrm{d}t < \infty.$$

Take K as before. Change μ such that $\mu(T(\mathbb{F}) \cap K) = 1$. Take $c_1, c_2 \in \mathbb{R}_{>0}$ as in Lemma 3.38. Then

$$\begin{aligned}
\int_{K \cap T(\mathbb{F})} |\chi(t) - 1|^{-\epsilon} \mathrm{d}t &\leq \sum_{s=0}^{\infty} q^{\epsilon \frac{s}{e}} \mu\left(T(\mathbb{F})_{\frac{s}{e}}\right) \leq \sum_{s=0}^{\infty} q^{\epsilon \frac{s}{e}} \frac{1}{[K : K_{\frac{s}{e}}]} \\
&\leq \sum_{s=0}^{\infty} q^{\epsilon \frac{s}{e}} C \left\lceil \frac{s}{e} \right\rceil^{c_1} q^{-\frac{1}{c_2} \frac{s}{e}} = \sum_{s=0}^{\infty} C \left\lceil \frac{s}{e} \right\rceil^{c_1} q^{\left(\epsilon - \frac{1}{c_2}\right) \frac{s}{e}},
\end{aligned}$$

where the second inequality is due to Lemma 3.39. The last sum converges if $\epsilon < \frac{1}{c_2}$. $\square$

From now on, T is a maximal $\mathbb{F}$-torus in G. Let $R(G, T)$ be the roots of T and G. Define

$$M := \max_{\alpha \in R(G,T)} \max_{i=1}^{m} \langle \alpha, X_i \rangle .$$

Corollary 3.40. *Let $\alpha \in R(G, T)$. The function $|\alpha(t) - 1|^{-\epsilon}$ is locally summable on $T(\mathbb{F})$ for $\epsilon < \frac{1}{M[\mathbb{E}:\mathbb{F}]}$.*

Proof. By the proof of Proposition 3.37, if $\epsilon < \frac{1}{c_2}$ for the c_2 of Lemma 3.38, the function $|\alpha(t) - 1|^{-\epsilon}$ is locally summable. The c_2 of Lemma 3.38 is equal to m_h, where $h = \psi(f) - \psi(g)$ for the g and f of Lemma 3.38. Therefore, $m_h \leq \max(m_f, m_g)$. The proof of Lemma 3.36 shows that

$$f(x_{11}, \ldots, x_{mn}) = \prod_{i=1}^{m} f_i(x_{i1}, \ldots, x_{in}),$$
$$g(x_{11}, \ldots, x_{mn}) = \prod_{i=1}^{m} g_i(x_{i1}, \ldots, x_{in}).$$

Thus $m_f = \max_{i=1}^m m_{f_i}$ and $m_g = \max_{i=1}^m m_{g_i}$.
The functions f_i and g_i are such that

$$\alpha \circ N_{\mathbb{E}/\mathbb{F}} \circ X_i \left(\sum_{j=1}^{n} a_j \alpha_j \right) = \prod_{\sigma \in \mathrm{Gal}(\mathbb{E}/\mathbb{F})} g_\sigma(a_1, \ldots, a_n)^{z_{i,\sigma}} = \frac{f_i(a_1, \ldots, a_n)}{g_i(a_1, \ldots, a_n)},$$

with $z_{i,\sigma} = \langle \sigma^{-1} \cdot \alpha, X_i \rangle$. Therefore,

$$\max(m_{f_i}, m_{g_i}) \leq \sum_{\sigma \in \mathrm{Gal}(\mathbb{E}:\mathbb{F})} |z_{i,\sigma}| \leq [\mathbb{E} : \mathbb{F}]M,$$

since $\sigma^{-1} \cdot \alpha \in R(G, T)$ for all $\sigma \in \mathrm{Gal}(\mathbb{E} : \mathbb{F})$. Thus

$$m_h \leq \max(m_f, m_g) = \max(\max_{i=1}^{m} m_{f_i}, \max_{i=1}^{m} m_{g_i}) \leq [\mathbb{E} : \mathbb{F}]M. \qquad \square$$

Lemma 3.41. *Let X be a space with measure μ and let $f : X \to \mathbb{R}_{\geq 0}$ and $g : X \to \mathbb{R}_{\geq 0}$. Assume that $f^{-\epsilon}$ and $g^{-\epsilon}$ are locally summable if $0 < \epsilon < \epsilon_o$. Then $(fg)^{-\epsilon}$ is locally summable if $0 < \epsilon < \frac{\epsilon_o}{2}$.*

Proof. If $f^{-\epsilon}$ is locally summable for all $\epsilon < \epsilon_o$, then $(f^2)^{-\epsilon}$ is locally summable for all $\epsilon < \frac{\epsilon_o}{2}$. Thus $f^{-\epsilon}$ and $g^{-\epsilon}$ are locally square integrable for all $\epsilon < \frac{\epsilon_o}{2}$. Then $(fg)^{-\epsilon}$ is locally summable for all $\epsilon < \frac{\epsilon_o}{2}$. $\square$

Theorem 3.42. *If $\epsilon < \frac{1}{2^{|R(G,T)|-1} M[\mathbb{E}:\mathbb{F}]}$, then $|D(t)|^{-\epsilon}$ is locally summable on T. Moreover, if $\epsilon < \frac{1}{2^{|R(G,T)|} M[\mathbb{E}:\mathbb{F}]}$, then, for all $n \in \mathbb{Z}_{\geq 0}$, the function $\mathrm{sd}(\gamma)^n |D(t)|^{-\epsilon}$ is locally summable on T.*

Proof. That $|D(t)|^{-\epsilon}$ is locally summable on T for $0 < \epsilon < \frac{1}{2^{|R(G,T)|-1} M[\mathbb{E}:\mathbb{F}]}$ follows from Corollary 3.40 and Lemma 3.41.

We show that sd_α^n is locally summable for all $n \in \mathbb{Z}_{\geq 0}$.
Let $t_o \in T$. If $\alpha(t_o) \neq 1$, then sd_α is locally constant near t_o and hence sd_α^n is locally summable around t_o.
If $\alpha(t_o) = 1$, then let $U := \alpha^{-1}(\mathcal{O})$ be a neighborhood of t_o. So it is enough to show that sd_α^n is locally summable in U.

By Proposition 3.37, there is an $\epsilon > 0$ such that $|\alpha(t) - 1|^{-\epsilon}$ is locally summable on T. Since $|\alpha(t) - 1|^{-1} = q^{\mathrm{sd}_\alpha(t)}$ if $\nu(\alpha(t) - 1) \geq 0$, there is $N \in \mathbb{N}$ such that

$\mathrm{sd}_\alpha(t)^n \leq N|\alpha(t)-1|^{-\epsilon}$ for all $t \in U$. Thus $\mathrm{sd}_\alpha(t)^n$ is locally summable on U, since $N|\alpha(t)-1|^{-\epsilon}$ is.

If $0 < \epsilon < \frac{1}{2^{|R(G,T)|}M[\mathbb{E}:\mathbb{F}]}$, then $|D(t)|^{-2\epsilon}$ is locally summable by the first statement of this Theorem. Since $\mathrm{sd}_\alpha(t)^n$ is locally summable for all $n \in \mathbb{Z}_{\geq 0}$, also $\mathrm{sd}(t)^{2n}$ is locally summable for all $n \in \mathbb{Z}_{\geq 0}$. Thus $\mathrm{sd}(t)^n|D(t)|^{-\epsilon}$ is locally summable for $0 < \epsilon < \frac{1}{2^{|R(G,T)|}M[\mathbb{E}:\mathbb{F}]}$, because $\mathrm{sd}(t)^{2n}$ and $|D(t)|^{-2\epsilon}$ are locally summable. □

In the case that char $\mathbb{F} = 0$, Harish-Chandra proved the existence of an $\epsilon > 0$ such that $|D(t)|^{-\epsilon}$ is locally summable in [HC70, Lemma 43].

3.6.2 Local summability of the character

Lemma 3.43. *Let $\omega \subset G$ be compact and T a maximal torus. Then ${}^G\omega \cap T$ is contained in a compact subset of T, i.e., it is bounded.*

Proof. Assume first that T is $\mathbb{F}$-split.
Let $d : G \to \mathbb{R}$ be the displacement function of $\mathcal{B}_e$.
CLAIM: For each $r \in \mathbb{R}$ the set $\{t \in T \mid d(t) \leq r\}$ is bounded.
By the proof of [BT72, Proposition 7.4.25], there is a retraction $\rho : \mathcal{B}_e \to \mathbb{A}_e$ defined by $y = u\rho(y)$ for some $u \in U^+$. Now ρ is T-equivariant:
$tux = tut^{-1}tx$, so $\rho(tux) = tx = t\rho(ux)$. Thus

$$d(x, tx) = d(\rho(ux), \rho(tux)) \leq d(ux, tux).$$

Thus $d(t) = d(tx, x)$ for $x \in \mathbb{A}_e(T)$. Therefore, $d(t) = d(\nu(t), \mathsf{O})$. Since there are only finitely many points $x \in T\mathsf{O}$ with $d(x, \mathsf{O}) \leq r$, the set $\{t \in T \mid d(t) \leq r\}$ is bounded.

The function $g \mapsto d(g)$ is a continuous class function, see [Moy00] and [DeB02b, Lemma 3.4.7]. Thus the image of ${}^G\omega$ is compact in $\mathbb{R}$. So ${}^G\omega \cap T$ is bounded.

Now we go to the general case:
Let $\mathbb{E}$ be a field extension of $\mathbb{F}$ such that $\mathcal{T}$ is $\mathbb{E}$-split. Since ${}^{\mathcal{G}(\mathbb{E})}\omega \cap \mathcal{T}(\mathbb{E})$ is bounded and ${}^G\omega \cap T \subset {}^{\mathcal{G}(\mathbb{E})}\omega \cap \mathcal{T}(\mathbb{E})$, also ${}^G\omega \cap T$ is bounded. □

If ω is compact modulo $Z(G)$, then ${}^G\omega \cap T$ is also compact modulo $Z(G)$. This could be proven in the same way as Lemma 3.43 by the displacement function on the reduced building. There is in this case a more elementary proof using $g \mapsto \det(\mathrm{Ad}(g) - x)$, see [HC70, Lemma 39].

Proposition 3.44. *Let T be a maximal torus of G containing a maximal split torus S. The function $\gamma \mapsto (\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(\gamma) + 1)^m |D(\gamma)|^{-\frac{1}{2}-\epsilon}$ is locally summable on ${}^GT$ for small $\epsilon \geq 0$.*

Proof. (See [HC70, VII,§1]) Let $\omega \subset G$ be compact open. By the Weyl integration formula and Proposition 3.34:

$$\begin{aligned}
&\int_{{}^GT} 1_\omega(g)(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(g)+1)^m|D(g)|^{-\frac{1}{2}-\epsilon}\,\mathrm{d}g \\
&\quad = |W|^{-1}\int_T |D(t)| \int_{T\backslash G} 1_\omega(g^{-1}tg)(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(g^{-1}tg)+1)^m|D(g^{-1}tg)|^{-\frac{1}{2}-\epsilon}\,\mathrm{d}g\mathrm{d}t \\
&\quad = |W|^{-1}\int_T |D(t)| \int_{T\backslash G} 1_\omega(g^{-1}tg)(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(t)+1)^m|D(t)|^{-\frac{1}{2}-\epsilon}\,\mathrm{d}g\mathrm{d}t \\
&\quad \leq C\int_T 1_\Omega(t)|D(t)|^{\frac{1}{2}}(\mathrm{ht}(\tilde{\Phi})\mathrm{sd}(t)+1)^{n+m}|D(t)|^{-\frac{1}{2}-\epsilon}\,\mathrm{d}t,
\end{aligned}$$

where $\Omega \subset T$ is compact and ${}^G\omega \cap T \subset \Omega$ (see Lemma 3.43). The right-hand side is finite by Theorem 3.42. □

Theorem 3.45. *Let (ρ, V) be a G-representation of finite length with character θ and $f \in C_c^\infty(G)$. Then for every torus T containing a maximal split torus S:*

$$\int_{{}^GT} f(g)\theta(g)\mathrm{d}g < \infty.$$

Proof. This follows from Propositions 3.19 and 3.44. □

The following Corollary has already been proven by Van Dijk in [Dij72, Theorem 3] by other means.

Corollary 3.46. *Assume that G is quasi-split. Let χ be a representation of $T := Z_G(S)$ of finite length. Then the character of $\mathrm{ind}_B^G(\chi)$ is locally summable.*

Proof. Since G is quasi-split, T is a maximal torus. Let γ be a regular semisimple element not in ${}^GT$. Let $K < G$ be a compact open subgroup such that $\gamma K \subset {}^GZ_G^0(\gamma)$ (see [MS12, Lemma 6.5] for a specific K). Let $K_o := K \cap \gamma K \gamma^{-1}$, then $K_o\gamma K_o \subset \gamma K \subset {}^GZ_G^0(\gamma)$. Thus ${}^GT \cap K_o\gamma K_o = \emptyset$. So for every open compact subgroup $K' \subset K_o$ we have $TgK' \cap TgK'\gamma K' = \emptyset$ for all $g \in G$. Since the character of $\mathrm{ind}_B^G\chi$ is supported on ${}^GT$,

$$\mathrm{tr}(\mathrm{ind}_B^G(\chi)(e_{K'} * \gamma * e_{K'})) = 0.$$

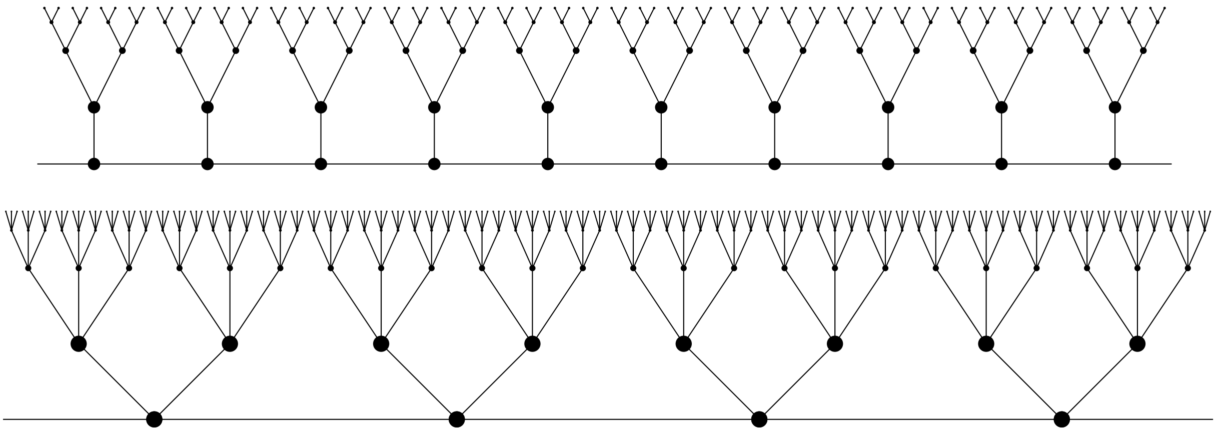
Hence the character of the induced representation is zero on the regular semisimple elements outside ${}^GT$. Now apply Theorem 3.45. □

3.7 GL_2: an overview

In this section we consider the group GL_2. Since this is a group with a root system of height one, we will be able to say something about the fixed points of non-split elements. Although this is one of the easiest examples of a reductive group, even in this case one can see the differences between fields of characteristic 0 and fields of positive characteristic (to be more precise characteristic 2). There is some overlap between the results of this section and the other results in the previous sections. The proofs of the results in this section are elementary, due to the simple structure of GL_2.

3.7.1 The Bruhat-Tits building of GL_2

The building of $GL_2(\mathbb{F})$ is a tree with infinitely many vertices and each vertex belongs to $q+1$ edges. The apartment is a line with infinitely many vertices regularly spaced along it. From the view point of the apartment of a torus, the building for $q=2$ and for $q=3$ looks as follows:



The line on the bottom is the apartment of a torus. The figure is not on scale: every edge of the graph should have length 1.

3.7.2 γ-Fixed points

Let γ be a regular semisimple element of G. Let T be the torus containing γ.

Assume T is $\mathbb{F}$-split. Then a point x is fixed by γ if and only if the distance from x to the apartment $\mathbb{A} = \mathbb{A}(T)$ is smaller or equal to $\mathrm{sd}(\gamma)$.

Assume T is an anisotropic torus. Then T splits over a 2-dimensional extension $\mathbb{E}$ over $\mathbb{F}$. Moreover, T can be identified with $\mathbb{E}^\times$ and $\mathbb{E}$ with $\mathbb{F}[\gamma]$. Let $\mathbb{A}$ be the apartment of $T(\mathbb{E})$ in $\mathcal{B}(\mathbb{E})$. Since T is anisotropic, $\mathbb{A}$ has only one Galois fixed point. Thus there exists at most one point in $\mathbb{A}$ that is also in the building over $\mathbb{F}$. Let x_0 be the point in the building over $\mathbb{F}$ closest to the apartment $\mathbb{A}$ in the building over $\mathbb{E}$. Let x be the point in $\mathbb{A}$ which is closest to x_0. Since $\mathbb{A}$ is T-invariant, x_0 must be a T-fixed point. The point x_0 must be a vertex in $\mathcal{B}(\mathbb{E})$, otherwise the segment (x_0, x) contains a point in $\mathcal{B}(\mathbb{F})$. Now we distinguish cases where $\mathbb{E}$ is unramified and $\mathbb{E}$ is totally ramified.

If $\mathbb{E}:\mathbb{F}$ is unramified, then $x_0 = x$ and x_0 is a vertex in $\mathcal{B}(\mathbb{F})$. Therefore, the number of vertices fixed by γ is equal to $1 + (q+1)\sum_{i=0}^{\mathrm{sd}(\gamma)-1} q^i = 1 + \frac{(q+1)(q^{\mathrm{sd}(\gamma)}-1)}{q-1}$ if $\mathrm{sd}(\gamma) \geq 0$.

If $\mathbb{E}:\mathbb{F}$ is totally ramified, then not necessarily $x_0 = x$. It turns out that x_0 is the middle of a chamber in $\mathcal{B}(\mathbb{F})$. The number of vertices y with $d(x_0, y) \leq \mathrm{sd}(\gamma)$ is equal to $2\sum_{i=0}^{\mathrm{sd}(\gamma)-\frac{1}{2}} q^i = \frac{2(q^{\mathrm{sd}(\gamma)+\frac{1}{2}}-1)}{q-1}$ if $\mathrm{sd}(\gamma) > 0$. If $\mathrm{sd}(\gamma) = 0$, then no vertex is fixed by γ.

The next figures illustrate the three situations in case the residue field has characteristic 2. We take $\mathrm{sd}(\gamma) = 3$ for the $\mathbb{F}$-split and the unramified case. For the totally ramified case we take $\mathrm{sd}(\gamma) = 2.5$. In the non-split cases $x_0 = x$. In red the vertices of $\mathbb{A}$. In green the vertices that are on the edge of the area fixed by γ. Thus the γ-fixed points are all the points on a line between a red and a green point or, equivalently, all the points on a line between two green points.

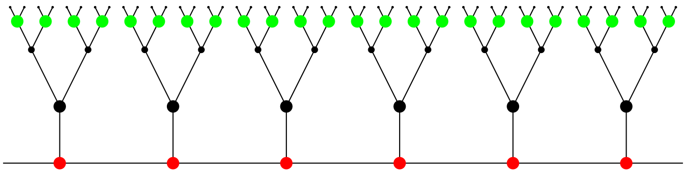


Figure 3.1: $\mathbb{F}$-split

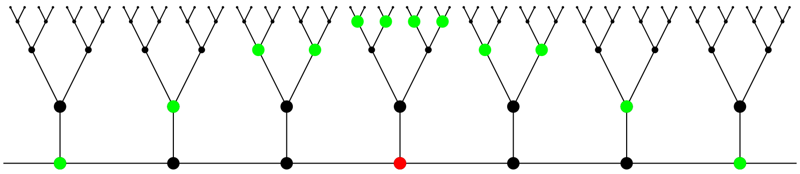


Figure 3.2: $\mathbb{E}$ is an unramified field extension of $\mathbb{F}$

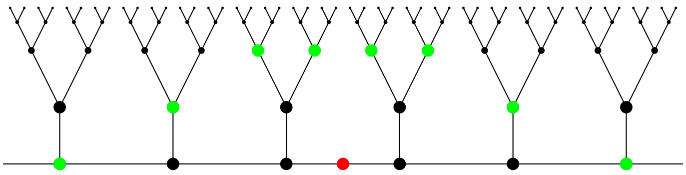


Figure 3.3: $\mathbb{E}$ is a totally ramified field extension of $\mathbb{F}$

3.7.3 The summability of $|D(\gamma)|^{-\frac{1}{2}}$

When char $\mathbb{F} = 0$, $|D(\gamma)|^{-\frac{1}{2}}$ is locally summable [HC70, Theorem 15]. We will see that $|D(\gamma)|^{-\frac{1}{2}}$ is locally summable around the identity if and only if char $\mathbb{F} \neq 2$. We first look at the case char $\mathbb{F} = 2$.

Assume $\mathbb{F} = \mathbb{F}_q((X))$, with q a power of 2. Define $K_i := 1 + X^i M_{2\times 2}(\mathbb{F}_q[[X]])$ for $i \geq 1$.

Lemma 3.47. *Let* $\gamma := \begin{pmatrix} 1+aX & bX \\ cX & 1+dX \end{pmatrix} \in K_1$, *then*

$$\nu(D(\gamma)) = 2\nu(a+d) + 2.$$

Hence if $\nu(D(\gamma)) = 2n$, *then for all* $\delta \in \gamma K_{n+1}$ *one has*

$$\nu(D(\delta)) = \nu(D(\gamma)) = 2n.$$

Proof. Let $\gamma := \begin{pmatrix} 1+aX & bX \\ cX & 1+dX \end{pmatrix}$. Now γ has characteristic polynomial

$$p_\gamma(x) = x^2 - (2 + aX + dX)x + (1+aX)(1+dX) - bcX^2.$$

The characteristic polynomial of a diagonal matrix with eigenvalues λ_1, λ_2 is

$$x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2.$$

So the difference between the two eigenvalues of γ is equal to $(a+d)X + 2 = (a+d)X$. Since γ is contained in a compact subgroup, λ_1, λ_2 are units in the ring of integers of $\mathbb{F}[\lambda_1]$. Thus

$$\nu(D(\gamma)) = \nu\left(\left(\frac{\lambda_1}{\lambda_2} - 1\right)\left(\frac{\lambda_2}{\lambda_1} - 1\right)\right) = 2\nu(\lambda_1 - \lambda_2) = 2\nu((a+d)X) = 2\nu(a+d) + 2.$$

So if $\nu(D(\gamma)) = 2n$, then $\nu(a+d) = n-1$. Thus if

$$\delta = \begin{pmatrix} 1+a'X & b'X \\ c'X & 1+d'X \end{pmatrix} \in \gamma K_{n+1},$$

then both $\nu(a-a')$ and $\nu(d-d')$ are greater than or equal to n. Thus $n-1 = \nu(a+d) = \nu(a'+d')$, so $\nu(D(\gamma)) = 2n$. $\square$

Define $C_n := \{\gamma \in K_1 \mid \nu(D(\gamma)) = 2n\}$. Let μ be the Haar measure on G such that $\mu(K_1) = 1$. The integral of $|D(\gamma)|^{-\frac{1}{2}}$ is:

$$\int_{K_1} |D(\gamma)|^{-\frac{1}{2}} \mathrm{d}\gamma = \sum_{n=1}^{\infty} q^n \mu(C_n).$$

Lemma 3.48. $\mu(C_n) = \frac{q-1}{q^n}$.

Proof. By the preceding lemma one has:

$$C_n = \bigcup_{\{\gamma\,:\,\gamma\in K_1/K_{n+1} | \nu(D(\gamma))=2n\}} \gamma K_{n+1}.$$

Notice that $\mu(K_{n+1}) = [K_1 : K_{n+1}] = q^{4n}$. So

$$\mu(C_n) = \frac{1}{q^{4n}} \#\{\gamma : \gamma \in K_1/K_{n+1} \mid \nu(D(\gamma)) = 2n\}.$$

Now $K_1/K_{n+1} \cong \left\{ \begin{pmatrix} 1+aX & bX \\ cX & 1+dX \end{pmatrix} : a,b,c,d \in \mathbb{F}_q[X]/X^n\mathbb{F}_q[X] \right\}$. The number of elements in $\{(a,b,c,d) \in (\mathbb{F}_q[X]/X^n\mathbb{F}_q[X])^4 \mid \nu(a+d) = n-1\}$ equals $q^{3n}(q-1)$ (for every a,b,c the number of d's with the properties: $a \equiv d \mod X^{n-1}$ and $a \not\equiv d \mod X^n$ is equal to $q-1$). Therefore, one has $\#\{\gamma : \gamma \in K_1/K_{n+1} \mid \nu(D(\gamma)) = 2n\} = q^{3n}(q-1)$. So $\mu(C_n) = \frac{1}{q^{4n}} \#\{\gamma : \gamma \in K_1/K_{n+1} \mid \nu(D(\gamma)) = 2n\} = \frac{q-1}{q^n}$. □

Theorem 3.49. *The function $|D(\gamma)|^{-\frac{1}{2}}$ is not integrable on K_1.*

Proof.

$$\int_{K_1} |D(\gamma)|^{-\frac{1}{2}} \mathrm{d}\gamma = \sum_{n=1}^{\infty} q^n \mu(C_n) = \sum_{n=1}^{\infty} (q-1) = \infty.$$ □

Since K_1 is compact, this theorem implies that $|D(\gamma)|^{-\frac{1}{2}}$ is not locally summable. A similar calculation shows that $|D(\gamma)|^{-\frac{1}{2}}$ is not integrable on K_i, for all $i \geq 0$. Therefore, $|D(\gamma)|^{-\frac{1}{2}}$ is not locally summable around the identity.

In contrast, $|D(\gamma)|^{-\frac{1}{2}-\epsilon}$ is integrable on $K_1 := 1 + \varpi M_{2\times 2}(\mathcal{O})$ when char $\mathbb{F} \neq 2$. Assume $\mu(K_1) = 1$.

Lemma 3.50. *If char $\mathbb{F} \neq 2$, then $|D(\gamma)|^{-\frac{1}{2}-\epsilon}$ is locally summable around 1 for $\epsilon < \frac{1}{2}$.*

Proof. Let $\gamma := \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}$. Now γ has characteristic polynomial

$$p_\gamma(x) = x^2 - (2+a+d)x + (1+a)(1+d) - bc.$$

The discriminant of this polynomial is equal to

$$(2+a+b)^2 - 4((1+a)(1+d) - bc) = (a-d)^2 - 4bc.$$

So the difference between the two eigenvalues is the square root of $(a-d)^2 - 4bc$. Thus $|D(\gamma)| = q^{\nu((a-d)^2-4bc)}$.

Let $B_n := \{\gamma \in K_1 \mid \nu(D(\gamma)) = n\}$.

The number of elements in $\{(a,b,c,d) \in (\mathcal{O}/\varpi^{n+1}\mathcal{O})^4 \mid \nu(a-d)^2 - 4bc)) = n\}$ is bounded by q^4 times the number of elements in $\{(a,b,c,d) \in (\mathcal{O}/\varpi^n\mathcal{O})^4 \mid \nu((a-d)^2 - 4bc)) \geq n\}$. Fix $a,b,d \in \mathcal{O}/\varpi^n\mathcal{O}$. The number of $c \in \mathcal{O}/\varpi^n\mathcal{O}$ such that

$$(a-d)^2 \equiv 4bc \mod \varpi^n$$

is bounded by $q^{\nu(b)}$. The number of $b \in \mathcal{O}/\varpi^n\mathcal{O}$ with $\nu(b) = k \leq n$ is bounded by q^{n-k}. Thus the number of elements in the set is bounded by $(n+1)q^{3n}$. Hence $\mu(B_n) \leq q^4(n+1)q^{-n}$. Thus

$$\int_{K_1} |D(\gamma)|^{-\frac{1}{2}-\epsilon} \mathrm{d}\gamma = \sum_{n=0}^{\infty} q^{(\frac{1}{2}+\epsilon)n} \mu(B_n) \leq q^4 \sum_{n=0}^{\infty} (n+1) q^{(\frac{1}{2}+\epsilon-1)n}.$$

So the integral is finite if $\epsilon < \frac{1}{2}$. □

The fact that $|D(\gamma)|^{-\frac{1}{2}}$ is not locally summable for fields of characteristic 2 is caused by the cancellation of $4bc$ in the expression $D(\gamma) = (a-b)^2 - 4bc$. Therefore only in the characteristic 2 case $D(\gamma)$ is always a square. For fields of odd characteristic, the expression for $D(\gamma)$ is an irreducible polynomial in the variables a, b, c, d of degree 2.

Although the integral over $|D(\gamma)|^{-\frac{1}{2}}$ does not exist in characteristic 2, the integral over the set of elements that are conjugate to elements in the split torus S does, see Proposition 3.44. For GL_2 we can show this by explicit calculations. Let $C := \{k \in K_1 \mid k \text{ is conjugate to an element in } S\}$.

Lemma 3.51. *The function $|D(\gamma)|^{-\frac{1}{2}}$ is integrable on C.*

Proof. Let $C_n := \left\{ k := \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \middle| \nu(a+d) = n \text{ and } k \in C \right\}$.
If $\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \in C_n$, then, since it is diagonalizable, the characteristic polynomial $x^2+(a+d)x+(a+d)+ad+bc+1$ has roots in $\mathbb{F}_q[X]$. Since $\nu(a+d) = n$, the characteristic polynomial becomes $x^2+ad+bc+1 \equiv (x+a+1)^2+bc \mod X^n$. Thus $bc \equiv (x+a+1)^2$ must be a square modulo X^n.
The number of $(b, c) \in (\mathcal{O}/X^n\mathcal{O})^2$ such that bc is a square modulo X^n is bounded by $nq^{\frac{n}{2}+n}$, since there are at most $q^{\frac{n}{2}}$ squares modulo X^n (this is typical for characteristic 2, for other characteristics the number of squares is greater than $\frac{q-1}{2}q^{n-1}$). Therefore:

$$\int_{C_n \cap K_1} |D(\gamma)|^{-\frac{1}{2}} \mathrm{d}\gamma \leq q^n \frac{q^n q^{\frac{n}{2}+n}}{q^{4n}} = nq^{-\frac{n}{2}}.$$

Thus $\int_C |D(\gamma)|^{-\frac{1}{2}} \mathrm{d}\gamma \leq \sum_{n=0}^{\infty} nq^{-\frac{n}{2}} < \infty$. □

3.8 Future work

This section is based on the study of fixed points in the reduced and extended building of compact regular semisimple elements in the centralizer of a maximal split torus. The understanding of the distribution of these fixed points gives the estimates for the character of an admissible smooth representation and the Weyl integration formula. We saw that both upper bounds are small enough to prove that the character of a finite length representation is locally summable on ${}^GT$, for T containing a maximal split torus.

A study of fixed points for general regular semisimple elements should lead to similar estimates. We hope that these upper bounds can be chosen small enough to prove that for every maximal torus T the character is locally summable on ${}^{G}T$. In the case that there are finitely many conjugacy classes of tori, the local summability of the character follows from the local summability on ${}^{G}T$. However, in positive characteristic, there could be infinitely many conjugacy classes of tori. In that case, the estimates should be synchronized in some way.

In §3.6.1 we introduced a generalization of the norm map $N_{\mathbb{E}/\mathbb{F}} : T(\mathbb{E}) \to T(\mathbb{F})$. It would be interesting to see whether this map has analogous properties as the regular norm map. In particular, whether the norm map is open and whether $[T(\mathbb{F}) : N_{\mathbb{E}/\mathbb{F}}(T(\mathbb{E}))] < \infty$.

Chapter 4

Nilpotent Orbits

This chapter is based on the paper [Wit17].

4.1 Nilpotent orbits and HC-theorem

Let $X \in \mathfrak{g}$, the Lie algebra of G. Then X is called *nilpotent*, when it is, viewed as a derivation of $k[G]$, a nilpotent linear map of $k[G]$. When G is a closed subgroup of $GL(V)$, this is equivalent with X acts nilpotent on V. A *nilpotent orbit* in $\mathfrak{g}$ is an $\mathrm{Ad}(G)$-orbit consisting of nilpotent elements.

The motivation to study nilpotent orbits was to get a proof of HC-theorem for the classical groups with a base field with low characteristic, say $p = 2$. The starting point was the proof of Rodier [Rod85] and Lemaire [Lem96] of HC-theorem for GL_n and DeBacker [DeB02a] for p-adic groups defined over fields with relatively high characteristic. The three cited papers mimic the proof of Harish-Chandra [HC99] of HC-theorem in the characteristic 0 case. To prove the HC-theorem, Rodier and Lemaire had to prove and DeBacker had to assume certain properties involving nilpotent orbits in the Lie algebra of the reductive group. To illustrate the use of nilpotent orbits, we give the main steps in the proof in [HC99]. This is based on the preface and introduction of [HC99].

For simplicity, we only bother about the local summability of θ_π around the identity of G. First, the problem is translated to the Lie algebra $\mathfrak{g}$ of G by the exponential map. Then it is shown that around 0 $\theta_\pi \circ \mathrm{Exp}$ is equal to a function η which Fourier transform is supported on $\mathrm{Ad}(G)\omega$, for some compact open $\omega \subset \mathfrak{g}$. Now functions of the form η are locally a linear combination of Fourier transforms of nilpotent orbital integrals. Since Fourier transforms of nilpotent orbital integrals are locally summable, so is η. Therefore, θ_π is locally summable.

When one tries to follow this strategy for reductive p-adic groups over a field with positive characteristic, one runs into a couple of problems. Of course, the exponential map is not defined in this case. However, there are also issues on the Lie algebra. Firstly, not all Lie algebras have a G-invariant nondegenerate bilinear form, which is necessary to define a Fourier transform (for example SL_n, with char $\mathbb{F} \mid n$). Secondly, although the author thinks nilpotent orbital integrals should be well-defined, he did not find a proof in the literature. Thirdly, in the proof one uses the finiteness of the number of nilpotent orbits in $\mathfrak{g}$, which does not hold in general. Finally, one of the auxiliary theorems in the proof is Howe's conjecture, which also does not hold in general. We will mainly focus on the finiteness of the number of nilpotent orbits and Howe's conjecture.

Although Howe's conjecture is not a necessary condition for the summability of θ_π (see e.g. [Lem05]), it is quite hard to do without. Also in [Lem05] Lemaire goes to GL_n to prove summability of θ_π for SL_n. Thus Lemaire goes to a central extension of the group for which Howe's conjecture holds. One of the consequences of this chapters is that such an approach is not available for $\mathbb{F}$-split reductive groups with bad characteristic, see Theorem 4.26.

In this chapter, we investigate Howe's conjecture, the geometry of the nilpotent orbits in $\mathfrak{g}$ and the relation between the nilpotent orbits and Howe's conjecture.

For $\omega \subset \mathfrak{g}$ define $J(\omega)$ to be the set of G-invariant distributions with support contained in the closure of ${}^G\omega = \mathrm{Ad}(G)\omega$. For an $\mathcal{O}$-lattice L in $\mathfrak{g}$ define $J_L(\omega)$ to be the image of $J(\omega)$ in the distributions of $\mathfrak{g}/L$ under the canonical map $\phi_L : \mathfrak{g} \to \mathfrak{g}/L$.

Conjecture 4.1 (Howe). *For all compact $\omega \subset \mathfrak{g}$ and all $\mathcal{O}$-lattices L of $\mathfrak{g}$:*

$$\dim J_L(\omega) < \infty.$$

This conjecture was proved by Howe in [How74] for $G = GL_n(\mathbb{F})$. Later, it was proved by Harish-Chandra, see [HC99], for general G in the case that char $\mathbb{F} = 0$. The main results on Howe's conjecture in this chapter are:

- Howe's conjecture holds for groups which split over a tamely ramified field extension and for which all nilpotent orbits are separable.
- For a split reductive group G, Howe's conjecture holds if and only if the characteristic is good for G and the adjoint map $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ is separable.

Thus, in particular, Howe's conjecture fails for a split reductive group for which the characteristic is bad. Rather surprisingly, there are reductive groups for which Howe's conjecture holds and which have infinitely many nilpotent orbits. Probably, there are only finitely many nilpotent orbits with a non-empty intersection with every neighborhood of 0, see for example in $SO_3(\mathbb{F})$ with char $\mathbb{F} = 2$ (Corollary 4.38). To prove Howe's conjecture for certain groups, we will just adapt the proof in [HC99].

Regarding the nilpotent orbits, we will show that if a split reductive group has finitely many nilpotent orbits, then all nilpotent orbits are separable. The converse statement has been proven in [McN04].

To be more precise, in this chapter we investigate the relation between the following statements for G (we will clarify the first six statements in §4.2):

1. p is good.
2. p is very good.

3. p does not divide the virtual number of components of $Z(G)$.
4. p does not divide the virtual order of $\pi_1(G_{\text{der}})$.
5. All the nilpotent orbits are separable.
6. The regular nilpotent orbit is separable.
7. The number of nilpotent orbits is finite.
8. Howe's conjecture holds for G.

If char $\mathbb{F} = 0$ (including $\mathbb{F} = \mathbb{C}, \mathbb{R}$), then all these statements hold for G. In case $\mathbb{F}$ has positive characteristic, these statements depend on G and p.

For general G, we will prove the following implications

$$\begin{tikzcd}
 & & & & (7) \\
(2) \arrow[r, "\text{Cor. 4.13}"] & (1)+(3)+(4) \arrow[r, leftrightarrow, "\text{Thm 4.52}"] \arrow[d] & (6) \arrow[r, leftrightarrow] & (5) \arrow[ur, "\text{[McN04]}"] \arrow[dr, "\text{Thm 4.33}(G\text{ t.r.})"] & \\
 & (1)+(3) \arrow[rrr, leftrightarrow, dashed, "?"] & & & (8)
\end{tikzcd}$$

where (5) implies (8) under the assumption that G splits over a tamely ramified field extension of $\mathbb{F}$. The question whether (5) implies (8) without the tameness condition on G is still open. The question if (7) and (5) are equivalent and the question if $(1) + (3)$ is equivalent to (8) are still open.
Moreover, if G is $\mathbb{F}$-split, then we get the following implications:

$$\begin{tikzcd}
 & & (2) \arrow[d] & \\
(5) \arrow[r, leftrightarrow] & (6) \arrow[r, leftrightarrow] & (1)+(3)+(4) \arrow[r, leftrightarrow, "\text{Thm 4.66}"] \arrow[d] & (7) \\
 & & (1)+(3) \arrow[r, leftrightarrow, "\text{Thm 4.49}"] & (8)
\end{tikzcd}$$

Besides the proofs of these implications, we will also give counterexamples for the non-implications. That $(1)+(3)$ does not imply (4) can be seen by the example PGL_p. That $(1)+(3)+(4)$ does not imply (2) can be seen by the example GL_p. That $(1)+(4)$ does not imply (3) can be seen by the example SL_p. That $(3)+(4)$ does not imply (1) can be seen in the simple groups of exceptional type. Thus for $\mathbb{F}$-split groups we have determined all the implications and non-implications between every possible combination of these 8 properties.

The first 4 statements are related to p and the root datum of G and the last 4 statements are related to the adjoint action of G on its Lie algebra $\mathfrak{g}$. The proofs of the implications from a collection of statements about the root datum to a statement about the adjoint action are mostly based on known proofs in the case that $\mathbb{F}$ has characteristic 0. The proofs of the implications from a statement about the adjoint action to a collection of statements about the root datum are different. In this case, we assume that one of the statements about the root datum does not hold and then show that the statement about the adjoint action does not hold. For example, we will show that $\neg(7)$ is a consequence of $\neg(1)$ or $\neg(3)$ or $\neg(4)$. The strategy is to make a surjective function from a part of the regular nilpotent elements of the Lie algebra to $\mathbb{F}/\mathbb{F}^{(p)}$ or $\mathbb{F}^\times/(\mathbb{F}^\times)^p$, which is G-invariant. For example, in $SL_2(\mathbb{F})$ with char $\mathbb{F} = 2$ we take the function

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mapsto x \mod (\mathbb{F}^\times)^2.$$

The proof that $\neg(1)$ or $\neg(3)$ implies $\neg(8)$ is based on the existence of such functions.

In the following table we list the properties of some $\mathbb{F}$-split groups. In the column nHwC are the set of primes P such that Howe's conjecture does not hold for $G(\mathbb{F})$ if and only if char $\mathbb{F} \in P$. In the column INO are the set of primes P such that the group $G(\mathbb{F})$ has infinitely many nilpotent orbits if and only if char $\mathbb{F} \in P$.

G	bad p	$\kappa_v(G)$	$\rho_v(G)$	nHwC	INO
GL_n	–	1	1	–	–
SL_n	–	n	1	$p\|n$	$p\|n$
PGL_n	–	1	n	–	$p\|n$
SO_{2n+1}	2	1	2	2	2
SO_{2n}	2	2	2	2	2
Sp_{2n}	2	2	1	2	2
F_4	$2,3$	1	1	$2,3$	$2,3$
G_2	$2,3$	1	1	$2,3$	$2,3$
E_8	$2,3,5$	1	1	$2,3,5$	$2,3,5$

Here $n \in \mathbb{N}_{\geq 2}$.

The obvious direction for generalizing the theory about Howe's conjecture and on the (in)finiteness of nilpotent orbits of this article is to look at reductive groups that are not $\mathbb{F}$-split. The proofs in the chapter depend heavily on the case by case consideration of the irreducible root systems. It would be nice to find unified proofs.

4.2 Notations

Unless otherwise stated, $\mathbb{F}$ is a local non-Archimedean field with uniformizer ϖ and ring of integers $\mathcal{O}$. We define $p :=$ char $\mathbb{F}$. For $n \in \mathbb{N}$ we define $\mathbb{F}^{(n)} := \{x^n : x \in \mathbb{F}\}$ and $\mathcal{O}^{(n)} := \{x^n : x \in \mathcal{O}\}$.

A prime number p is *bad* for a root system R if

1. $p = 2$ and R has a component not of type A_n.
2. $p = 3$ and R has a component of type E_n, F_4 or G_2.
3. $p = 5$ and R has a component of type E_8.

A prime number p is *good* for R if it is not bad. See [SS70, §4.1] for equivalent definitions of good primes.

A prime number p is *very good* for R if it is good and R does not have a component of type A_n with p a divisor of $n+1$.
A prime number p is *(very) good* for G if it is (very) good for the root system of G.

A G-orbit $\mathrm{Ad}(G)x$ in $\mathfrak{g}$ is called *separable* if one of the following equivalent conditions holds:

1. The differential of the map $g \mapsto \mathrm{Ad}(g)x$ is surjective.
2. $\dim\{g \in G \mid \mathrm{Ad}(g)x = x\} = \dim\{y \in \mathfrak{g} \mid [y,x] = 0\}$.
3. The Lie algebra of $\{g \in G \mid \mathrm{Ad}(g)x = x\}$ is equal to $\{y \in \mathfrak{g} \mid [y,x] = 0\}$.

4.2.1 $\kappa_v(G)$ & $\rho_v(G)$

Let T be a maximal torus of G. Let $R(G,T)$ be the roots of G relative to T and let $R^\vee(G,T)$ be the coroots. Let $X^*(T)$ be the characters of T and let $X_*(T)$ be the cocharacters of T. The two embeddings $R(G,T) \hookrightarrow X^*(T)$ and $R^\vee(G,T) \hookrightarrow X_*(T)$ induce group homomorphisms $\Phi : X_*(T) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G,T), \mathbb{Z})$ and $\Phi^\vee : X^*(T) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G,T)^\vee, \mathbb{Z})$. We call $\rho_v(G) := |\mathrm{coker}\, \Phi^\vee|$ the *virtual order of* $\pi_1(G_{\mathrm{der}})$. We call $\kappa_v(G) := |\mathrm{coker}\, \Phi|$ the *virtual number of components of* $Z(G)$. To explain the names of these numbers, we look at complex tori and complex reductive groups.

Lemma 4.2. *Let $\mathscr{T}$ and $\mathscr{S}$ be two complex tori and $\phi : \mathscr{T} \to \mathscr{S}$. Let $\phi^* : X^*(\mathscr{S}) \to X^*(\mathscr{T})$ be the map $\epsilon \mapsto \epsilon \circ \phi$. Then*

$$|\ker \phi / (\ker \phi)^o| = |(\mathrm{coker}\, \phi^*)_{\mathrm{tor}}|,$$

where $(\mathrm{coker}\, \phi^*)_{\mathrm{tor}}$ is the torsion part of the cokernel of ϕ^*.

Proof. Choose the bases $\delta_1, \ldots, \delta_m$ for $X^*(\mathscr{S})$ and $\epsilon_1, \ldots, \epsilon_n$ for $X^*(\mathscr{T})$ in such a way that

$$\begin{aligned} \phi^*(\delta_1) &= d_1\epsilon_1, \\ \vdots \; &= \; \vdots \\ \phi^*(\delta_k) &= d_k\epsilon_k, \\ \phi^*(\delta_{k+1}, \ldots, \delta_m) &\in \langle d_1\epsilon_1, \ldots, d_k\epsilon_k \rangle . \end{aligned}$$

Then $\prod_{i=1}^k d_i = |(\mathrm{coker}\, \phi^*)_t|$.
Thus

$$\begin{aligned} \ker \phi &:= \{t \in \mathscr{T} \mid \epsilon_i(t)^{d_i} = 1 \text{ for all } 1 \le i \le k\}, \\ (\ker \phi)^o &:= \{t \in \mathscr{T} \mid \epsilon_i(t) = 1 \text{ for all } 1 \le i \le k\}. \end{aligned}$$

Therefore

$$|\ker \phi / (\ker \phi)^o| = \left| \prod_{i=1}^k \mathbb{Z}/d_i\mathbb{Z} \right| = \prod_{i=1}^k d_i = |(\mathrm{coker}\, \phi^*)_{\mathrm{tor}}|.$$ □

Corollary 4.3. *For a complex reductive group $\mathscr{G}$, $|\mathrm{coker}\, \Phi| = |\pi_0(Z(\mathscr{G}))|$ and $|\mathrm{coker}\, \Phi^\vee| = |\pi_1(\mathscr{G}_{der})|$.*

Proof. Look at the adjoint map: $\mathrm{Ad} : \mathscr{G} \to \mathscr{G}^{\mathrm{ad}}$. Let $\mathscr{T}$ be a maximal torus of $\mathscr{G}$ and $\mathscr{T}^{\mathrm{ad}} = \mathrm{Ad}(\mathscr{T})$. Then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z}) = X_*(\mathscr{T}^{\mathrm{ad}})$ and Φ is the map corresponding with $\mathrm{Ad} : \mathscr{T} \to \mathscr{T}^{\mathrm{ad}}$:

$$\Phi(\epsilon) := \mathrm{Ad} \circ \epsilon.$$

We define $\Phi^{\mathrm{tr}} : X^*(\mathscr{T}^{\mathrm{ad}}) \to X^*(\mathscr{T})$ as follows:

$$\Phi^{\mathrm{tr}}(\epsilon) := \epsilon \circ \mathrm{Ad}.$$

The cokernel of Φ^{tr} has a torsion group of order $|\mathrm{coker}\, \Phi|$. Thus

$$|\mathrm{coker}\, \Phi| = |Z(\mathscr{G})/Z(\mathscr{G})^o| = |\pi_0(Z(\mathscr{G}))|,$$

since $\mathscr{T} \cap \ker \mathrm{Ad} = Z(\mathscr{G})$. Let $\mathscr{G}_{\mathrm{sc}}$ be the simply connected cover of $\mathscr{G}_{\mathrm{der}}$. Let $\pi : \mathscr{G}_{\mathrm{sc}} \to \mathscr{G}$ be the following morphism: $\mathscr{G}_{\mathrm{sc}} \twoheadrightarrow \mathscr{G}_{\mathrm{der}} \hookrightarrow \mathscr{G}$.
Let $\mathscr{T}_{\mathrm{sc}}$ be the maximal torus of $\mathscr{G}_{\mathrm{sc}}$ such that $\pi(\mathscr{T}_{\mathrm{sc}}) = \mathscr{T} \cap \mathscr{G}_{\mathrm{der}}$. Then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z}) = X_*(\mathscr{T}_{\mathrm{sc}})$ and $\Phi^\vee$ is the map corresponding with $\pi : \mathscr{T}_{\mathrm{sc}} \to \mathscr{T}$. Thus

$$|\mathrm{coker}\, \Phi^\vee| = |\ker \pi| = |\pi_1(\mathscr{G}_{\mathrm{der}})|. \qquad \square$$

4.2.2 Chevalley basis

The first part of this subsection is based on [BT84, §3.2].
Let G be a $\mathbb{F}$-split reductive group and T a maximal torus. Let $\mathfrak{g}$ be the Lie algebra of G. Let $R := R(G, T)$ be the roots of G and T. Let R^+ be a set of positive roots of R and let Δ be the set of corresponding simple roots.

For $\beta \in R$ there are elements H_β and E_β in $\mathfrak{g}$, such that for all $\alpha, \beta \in R$:

$$\begin{aligned} &[H_\alpha, H_\beta] = 0, \\ &[H_\alpha, E_\beta] = \langle \alpha^\vee, \beta \rangle \, E_\beta, \\ &[E_\beta, E_\alpha] = \begin{cases} N_{\beta,\alpha} E_{\beta+\alpha} & \text{if } \beta + \alpha \in R, \\ H_\beta & \text{if } \alpha = -\beta, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where each $N_{\beta,\alpha} \in \mathbb{Z}$. For each $\beta \in R$ there exists a unique map $u_\beta : \mathbb{F} \to G$, such that $\mathrm{d}\beta(1) = E_\beta$ and for all $t \in T$ and $x \in \mathbb{F}$, $t u_\beta(x) t^{-1} = u_\beta(\beta(t)x)$. Then $\beta^\vee$, the coroot of β, is equal to

$$\beta^\vee(\lambda) = u_\beta(\lambda) u_{-\beta}(-\lambda^{-1}) u_\beta(\lambda) u_\beta(-1) u_{-\beta}(1) u_\beta(1).$$

Moreover, $\mathrm{d}\beta^\vee(1) = H_\beta$.
The set $\{H_\alpha : \alpha \in \Delta\} \cup \{E_\beta : \beta \in R\}$ is called a *Chevalley basis*. (The term "basis" is misplaced here, since if G is not semisimple it does not span $\mathfrak{g}$ and if $G = PGL_n$ and $\mathrm{char}\, \mathbb{F} \mid n$ it is not linearly independent, see Lemma 4.43. However, if the characteristic is 0 it is a basis for $\mathfrak{g}'$, the Lie algebra of $G_{\mathrm{der}} = (G, G)$. The E_β are always linearly independent.)

The adjoint representation $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ is determined by the following formulas:

$$\begin{aligned}
\mathrm{Ad}(u_\beta(\lambda))E_\alpha &= \begin{cases} E_\beta & \text{if } \beta = \alpha, \\ E_{-\beta} + \lambda H_\beta - \lambda^2 E_\beta & \text{if } \alpha = -\beta, \\ \sum_{i\geq 0} M_{\beta,\alpha,i}\lambda^i E_{i\beta+\alpha} & \text{otherwise,} \end{cases} \\
\mathrm{Ad}(t)E_\beta &= \beta(t)E_\beta, \\
\mathrm{Ad}(u_\beta(\lambda))H &= H - d\beta(H)\lambda E_\beta, \\
\mathrm{Ad}(t)H &= H,
\end{aligned}$$

for all $H \in \mathfrak{t}$, the Lie algebra of T, and constants $M_{\beta,\alpha,i} \in \mathbb{F}$.
The $\mathbb{F}$-points of the image of the algebraic map Ad will be denoted by $\mathrm{Ad}(G)$ or G^{ad}. From now on, we fix a Chevalley basis on $\mathfrak{g}$.

4.3 Regular nilpotent orbits

In the first part of this short introduction to nilpotent orbits, especially regular nilpotent orbits, we will follow [Car85, §5.1]. Although [Car85, §5.1] treats regular unipotent elements, we can easily adapt it to regular nilpotent elements.

For each $\alpha \in R$, define $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : \mathrm{Ad}(t)x = \alpha(t)x\}$. We define the height function $\mathrm{ht} : R \to \mathbb{Z}$ as follows:

$$\mathrm{ht}\left(\sum_{\alpha\in\Delta} c_\alpha \alpha\right) := \sum_{\alpha\in\Delta} c_\alpha.$$

For $z \in \mathbb{Z}$ we define the following subspaces of $\mathfrak{g}$:

$$\begin{aligned}
\mathfrak{n}_z &:= \bigoplus_{\alpha\in R|\mathrm{ht}(\alpha)=z} \mathfrak{g}_\alpha, \\
\mathfrak{n}_{\geq z} &:= \bigoplus_{\alpha\in R|\mathrm{ht}(\alpha)\geq z} \mathfrak{g}_\alpha.
\end{aligned}$$

A nilpotent element of $n \in \mathfrak{g}$ is called a *regular nilpotent element* if and only if

$$\dim Z_G(n) = \dim T.$$

Proposition 4.4. *Let $\mathbb{F}$ be an algebraically closed field. Let G be a connected reductive group (over $\mathbb{F}$). Then there exist regular nilpotent elements in $\mathfrak{g}$, and any two are conjugate. Let $n \in \mathfrak{g}$ be nilpotent. The following conditions on n are equivalent:*

1. *n is regular.*

2. *There is a unique Borel subgroup B of G such that n is in the Lie algebra of B.*

3. *n is conjugate to an element of the form $\sum_{\alpha\in R^+} \lambda_\alpha E_\alpha$ with $\lambda_\alpha \neq 0$ for all $\alpha \in \Delta$.*

Proof. We use the proof of [Car85, Proposition 5.1.2 & 5.1.3]. That there are only finitely many nilpotent classes is proven in [HS85, Theorem 1]. Then by the same proof of [Car85, Proposition 5.1.2] there exist regular nilpotent elements in $\mathfrak{g}$ and any two are conjugate. The U-orbit of n is closed, since every orbit of a unipotent group is closed [Ste74, Proposition 2.5]. Therefore, the proof of [Car85, Proposition 5.1.3] is also valid for nilpotent elements of $\mathfrak{g}$. □

Corollary 4.5. *Let n, n' be regular nilpotent elements of the Lie algebra of B. If $g \in G$ is such that $gng^{-1} = n'$, then $g \in B$.*
If $n = \sum_{\alpha\in\Delta} c_\alpha E_\alpha$ and $n' = \sum_{\alpha\in\Delta} d_\alpha E_\alpha$, then the following statements are equivalent:

1. *n and n' are conjugate by an element of $G(\mathbb{F})$.*

2. *n and n' are conjugate by an element of $T(\mathbb{F})$.*

3. *There is a $t \in T$ such that for all $\alpha \in \Delta$, $d_\alpha = \alpha(t)c_\alpha$.*

Proof. By Proposition 4.4, $B = gBg^{-1}$, since n' is in the Lie algebra of B and gBg^{-1}. Thus $g \in N_G(B) = B$.
Assume that $n = \sum_{\alpha\in\Delta} c_\alpha E_\alpha$ and $n' = \sum_{\alpha\in\Delta} d_\alpha E_\alpha$.
If n and n' are conjugate, then there exist $t \in T$ and $u \in U$ such that $\mathrm{Ad}(tu)n = n'$.
Since U acts trivially on $\mathfrak{n}/\mathfrak{n}_{\geq 1}$ and $\mathrm{Ad}(t)E_\alpha = \alpha(t)E_\alpha$, the second statement follows.
Let $t \in T$. Then $\mathrm{Ad}(t)n = n'$ if and only if for all $\alpha \in \Delta$, $d_\alpha = \alpha(t)c_\alpha$. □

Corollary 4.5 shows that ${}^G n \cap B = {}^B n$ for all regular $n \in \mathfrak{b}$.
Recall the definition of Φ:

$$\Phi : X_*(T) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G,T), \mathbb{Z})$$
$$\Phi : \gamma \mapsto (\alpha \mapsto \langle \gamma, \alpha \rangle).$$

The first reason for defining Φ is the following proposition. Recall that the virtual number of components of $Z(G)$ is $\kappa_v(G) := |\mathrm{coker}\,\Phi|$.

Proposition 4.6. *If G is $\mathbb{F}$-split and $p \mid \kappa_v(G)$, then there are infinitely many rational regular nilpotent orbits in $\mathfrak{g}(\mathbb{F})$.*

Proof. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for $R(G,T)$. Define the function $\epsilon_i \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G,T), \mathbb{Z})$ for $1 \leq i \leq n$ by:

$$\epsilon_i(\alpha_j) := \delta_{ij}.$$

So $\epsilon_1, \ldots, \epsilon_n$ is a basis for $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G,T), \mathbb{Z})$. Let L be the image of Φ. Take a compatible basis for $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G,T), \mathbb{Z})$ and L: $b_1, \ldots, b_n$ and $d_1 b_1, \ldots, d_n b_n$ with $d_i \mid d_{i+1}$. Since the cokernel is finite, it has $\prod_{i=1}^n d_i$ elements. Define the matrix $M \in GL_n(\mathbb{Z})$ by

$$b_i = \sum_{j=1}^{k} M_{ij}\epsilon_j.$$

Look at the following subset of $\mathfrak{n}_1$:

$$\mathfrak{n}_1' := \left\{ \sum_{\alpha\in\Delta} c_\alpha E_\alpha : c_\alpha \in \mathbb{F}^\times \right\}.$$

Define π to be the following parametrization of $\mathfrak{n}_1'$:

$$\pi : (\mathbb{F}^\times)^n \to \mathfrak{n}_1', \quad \pi(c_1, \cdots, c_n) := \sum_{i=1}^{n} c_i E_{\alpha_i}.$$

Since $\sum_{\alpha\in\Delta} c_\alpha E_\alpha$ is a regular nilpotent element, it is in the same conjugacy class of $G(\mathbb{F})$ as $\sum_{\alpha\in\Delta} d_\alpha E_\alpha$ if and only if there is $t \in T$ such that $d_\alpha = \alpha(t)c_\alpha$ for all $\alpha \in \Delta$, by Corollary 4.5.
Let $A \in GL_n(\mathbb{Z})$ and define $\phi_A : (\mathbb{F}^\times)^n \to (\mathbb{F}^\times)^n$ by:

$$\phi_A(x_1, \ldots, x_n) := \left(\prod_{i=1}^n x_i^{a_{1i}}, \ldots, \prod_{i=1}^n x_i^{a_{ni}} \right).$$

Now $\pi \circ \phi_A$ is also a parametrization of $\mathfrak{n}_1'$ and

$$\phi_A \pi^{-1}(t\pi(\phi_{A^{-1}}(x_1, \ldots, x_n))t^{-1}) = \left(\left(\prod_{i=1}^n \alpha_i(t)^{a_{1i}} \right) x_1, \ldots, \left(\prod_{i=1}^n \alpha_i(t)^{a_{ni}} \right) x_n \right).$$

Define the action of T on $(\mathbb{F}^\times)^n$ to be the action with respect to A.
Take $A := (M^{-1})^t$.
We claim that for every $\gamma \in X_*(T)$ with $\Phi(\gamma) = \sum_{i=1}^n z_i d_i b_i$ one has the following action on $(\mathbb{F}^\times)^n$ with respect to A of $\gamma(s)$:

$$(x_1, \ldots, x_n) \mapsto (s^{z_1 d_1} x_1, \ldots, s^{z_n d_n} x_n).$$

To prove this claim, consider the factor in front of x_j:

$$\prod_{i=1}^n \alpha_i(\gamma(s))^{a_{ji}} = s^{\sum_{i=1}^n a_{ji} \langle \gamma, \alpha_i \rangle}.$$

Evaluate the power of s:

$$\sum_{i=1}^n a_{ji} \langle \gamma, \alpha_i \rangle = \sum_{i=1}^n a_{ji} \sum_{k=1}^n z_k d_k b_k(\alpha_i) = \sum_{i=1}^n a_{ji} \sum_{k=1}^n z_k d_k m_{ki} = \sum_{k=1}^n z_k d_k \sum_{i=1}^n a_{ji} m_{ki}.$$

Since $A = (M^{-1})^t$, one has that $\sum_{i=1}^n a_{ji} m_{ki} = \delta_{jk}$. Therefore,

$$\sum_{i=1}^n a_{ji} \langle \gamma, \alpha_i \rangle = z_j d_j \text{ , hence}$$
$$\prod_{i=1}^n \alpha_i(\gamma(s))^{a_{ji}} = s^{z_j d_j}.$$

Since $p \mid \#\mathrm{coker}\,\Phi$, then $p \mid d_n$. Identify $\mathfrak{n}_1'$ with $(\mathbb{F}^\times)^n$ via the parametrization $\pi \circ \phi_{M^t}$. Look at the n-th coordinate: $x_n \mapsto s^{z_\gamma d_n} x_n$ for every pair $\gamma \in X_*(T)$, $s \in \mathbb{F}^\times$. The images of the cocharacters generate the torus, so the orbit of the n-th coordinate under T is contained in $\{s^{d_n} x_n : s \in \mathbb{F}^\times\}$. Hence if $(x_1, \ldots, x_n)$ is in the same orbit as $(y_1, \ldots, y_n)$, then there is $s \in \mathbb{F}^\times$ such that $s^{d_n} x_n = y_n$. Since $p \mid d_n$, the group $\mathbb{F}^\times / (\mathbb{F}^\times)^{d_n}$ is infinite. We conclude that if p divides the order of the cokernel, then there are infinitely many regular nilpotent orbits. □

4.4 The virtual number of components of $Z(G)$

As we saw in the previous section, when p divides the virtual number of components of $Z(G)$ there are infinitely many regular nilpotent orbits. In this section we show that even more properties that hold when the characteristic is zero, do not hold anymore when $p \mid \kappa_v(G)$. After giving these counterexamples for theorems that hold in characteristic zero, at the end of this section we show that $p \mid \kappa_v(G)$ for a restrictive class of reductive groups. By the way, the condition $p \mid \kappa_v(G)$ is based on the group $SL_n(\mathbb{F})$ with $p \mid n$. As it turns out in the end, for $p \geq 5$ every example has a factor of type A_{n-1}. In this section, all the properties are geometric in nature, so we do not have to worry about rationality.

4.4.1 Separability and $\kappa_v(G)$

Lemma 4.7. *Let $X := \sum_{\alpha\in\Delta} E_\alpha$. The map $[X,\cdot] : \mathfrak{t} \to \mathfrak{n}_1$ is not surjective if and only if $p \mid \kappa_v(G)$.*

Proof. Consider $\mathfrak{t}$ as $X_*(T) \otimes \mathbb{F}$. Let $X_1, \ldots, X_n$ be a basis for $X_*(T)$. Let $Y \in X_*(T)$, then $[\mathrm{d}Y(1), E_\alpha] = \langle \alpha, Y\rangle E_\alpha$. The matrix M corresponding to Φ with respect to the basis $X_1, \ldots, X_n$ and the dual basis of Δ in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z})$ is the same as the matrix corresponding to $[X, \cdot]$ with respect to the basis $X_1, \ldots, X_n$ and $(E_\alpha : \alpha \in \Delta)$. Let $d_1, \ldots, d_{|\Delta|}$ be the integers on the diagonal of the Smith normal form of M. Then $\kappa_v(G) = \#\mathrm{coker}\,\Phi = \prod_{i=1}^{|\Delta|} d_i$. Also there are $E_1, \ldots, E_{|\Delta|}$ such that $\mathfrak{n}_1 = \left\langle E_1, \ldots, E_{|\Delta|}\right\rangle$ and $[X, \mathfrak{t}] = \left\langle d_1E_1, \ldots, d_nE_{|\Delta|}\right\rangle$. □

Recall that an G-orbit $\mathrm{Ad}(G)x$ in $\mathfrak{g}$ is separable if and only if

$$\dim\{g \in G \mid \mathrm{Ad}(g)x = x\} = \dim\{y \in \mathfrak{g} \mid [y, x] = 0\}.$$

Theorem 4.8. *If $p \mid \kappa_v(G)$, then the regular orbit is not separable.*

Proof. Let T be a torus and $X \in \mathfrak{n}_1$ a regular element. By Proposition 4.4, then $\dim Z_T(X) = \dim T - |\Delta|$. Thus if the orbit of X is separable, then $[X, \cdot] : \mathfrak{t} \to \mathfrak{n}_1$ has a kernel of dimension $\dim T - |\Delta|$. Thus $[X, \cdot]$ must be surjective. Since $p \mid \kappa_v(G)$ the map $[X, \cdot]$ is not surjective. Hence the orbit of X is not separable. □

4.4.2 Ad and $\kappa_v(G)$

Let G be a reductive $\mathbb{F}$-group. Let $\mathrm{Ad}(G)$ be the image of the adjoint mapping $\mathrm{Ad} : G \to GL(\mathfrak{g})$. We will go back and forth between G and $\mathrm{Ad}(G)$. Therefore we have a look at the adjoint map $\mathrm{Ad} : G \to \mathrm{Ad}(G)$. The adjoint map is defined over $\mathbb{F}$. We will show that $\mathrm{d}(\mathrm{Ad})$ maps non-zero nilpotent elements to non-zero nilpotent elements. Ad is separable (i.e., $\mathrm{d}(\mathrm{Ad})$ is surjective) if and only if $p \nmid \kappa_v(G)$. To distinguish the objects associated with $\mathrm{Ad}(G)$ from the ones associated with G, the ones associated with $\mathrm{Ad}(G)$ get a superscript ad: G^{ad},$\mathfrak{g}^{\mathrm{ad}}$, $\mathfrak{n}^{\mathrm{ad}}$, and so on.

Lemma 4.9. $\mathrm{d}(\mathrm{Ad}) : \mathfrak{n} \to \mathfrak{n}^{ad}$ *is an isomorphism.*

Proof. Take a Chevalley basis on $\mathfrak{g}$. Let $\alpha \in R(G,T)$. Let G_α^{ad} be the image of $\mathrm{Ad} \circ u_\alpha : \mathbb{F} \to G^{\mathrm{ad}}$. The action of $u_\alpha(x)$ on certain elements of $\mathfrak{g}$ is as follows:

$$\begin{aligned}\mathrm{Ad}(u_\alpha(x))E_{-\alpha} &= E_{-\alpha} + x\mathrm{d}\alpha^\vee(1) - x^2 E_\alpha,\\ \mathrm{Ad}(u_\alpha(x))H &= H - d\alpha(H)xE_\alpha.\end{aligned}$$

Since $\langle \alpha, \alpha^\vee \rangle = 2$, either $\mathrm{d}\alpha^\vee(1) \neq 0$ or there exists $H \in \mathfrak{t}$ such that $\mathrm{d}\alpha(H) \neq 0$. Therefore, $\mathrm{Ad} \circ u_\alpha$ is an isomorphism between $\mathbb{F}$ and its image in G. Since $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $t \in T$ and $x \in \mathbb{F}$, also

$$t\mathrm{Ad}(u_\alpha(x))t^{-1} = \mathrm{Ad}(u_\alpha(\alpha(t)x))$$

for all $t \in T^{\mathrm{ad}}$ and $x \in \mathbb{F}$. Thus $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g}_\alpha \to \mathfrak{g}_\alpha^{\mathrm{ad}}$ is an isomorphism. Therefore $\mathrm{d}(\mathrm{Ad}) : \mathfrak{n} \to \mathfrak{n}^{\mathrm{ad}}$ is injective. Since $\dim \mathfrak{n} = \dim \mathfrak{n}^{\mathrm{ad}}$, the Lemma follows. $\square$

Proposition 4.10. *The map* $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ *is separable if and only if the characteristic of* $\mathbb{F}$ *does not divide the virtual number of components of* $Z(G)$.

Proof. Let Δ be a system of positive roots for $R(G,T)$.
Define $n := |\Delta|$ and let $\alpha_1, \ldots, \alpha_n$ be the roots in Δ. Take $\gamma_1, \ldots, \gamma_n \in X_*(T)$ such that the image of Φ is generated by $\gamma_1, \ldots, \gamma_n$. The number of elements in the cokernel of Φ is equal to the determinant of the matrix $M_{ij} := \langle \gamma_j, \alpha_i \rangle$. Since $\mathrm{d}(\mathrm{Ad})$ is surjective on $\mathfrak{n}_+^{\mathrm{ad}} \oplus \mathfrak{n}_-^{\mathrm{ad}}$, we only have to look whether $\mathrm{Ad} : T \to T^{\mathrm{ad}}$ is separable. Identify T^{ad} with a torus of dimension n in such a way that the map Ad is as follows:

$$t \mapsto \begin{pmatrix} \alpha_1(t) & & 0 \\ & \ddots & \\ 0 & & \alpha_n(t) \end{pmatrix}.$$

The Lie algebra of a torus S is canonically isomorphic to $X_*(S) \otimes_{\mathbb{Z}} \mathbb{F}$ by [Spr98, 4.4.11(4)]. With this isomorphism the map $\mathrm{d}(\mathrm{Ad})$ is the linear map such that for $\gamma \in X_*(T)$, $\mathrm{d}(\mathrm{Ad})(\gamma) = \mathrm{Ad} \circ \gamma$. Now the images of $\mathrm{Ad} \circ \gamma_1, \ldots, \mathrm{Ad} \circ \gamma_n$ generate the image of $\mathrm{d}(\mathrm{Ad})$. Thus the image of $\mathfrak{t}$ is generated by the vectors $\sum_{i=i}^n \langle \gamma_j, \alpha_i \rangle \chi_i$ for $j = 1, \ldots, n$. This is surjective if and only if the corresponding matrix has non-zero determinant. The corresponding matrix is equal of M. Thus $p \nmid \kappa_v(G) = \#\mathrm{coker}\,\Phi$ if and only if M is invertible if and only if Ad is separable. $\square$

4.4.3 Very good primes and $\kappa_v(G)$

Lemma 4.11. *If* $Y \subset X_*(T)$ *such that* $\Phi(Y)$ *has finite index in* $\mathrm{Hom}(\mathbb{Z}R(G,T), \mathbb{Z})$*, then* $\#\mathrm{coker}\,\Phi$ *divides* $\#\mathrm{coker}\,\Phi|_Y$.

Proof. The lemma follows from general abstract non-sense:

$$\begin{tikzcd} A \arrow[dr, "f"] \arrow[dd, hook, "\iota"'] & & \mathrm{coker}\, g \\ & B \arrow[ur, "cg"] \arrow[dr, "cf"'] & \\ C \arrow[ur, "g"] & & \mathrm{coker}\, f \arrow[uu, dotted, two heads, "h"'] \end{tikzcd}$$

Since $cg \circ g \circ \iota = 0$ there is a unique morphism $h : \operatorname{coker} f \to \operatorname{coker} g$ such that $cg = cf \circ h$. Since cg is surjective, h is also surjective. Thus $\#\operatorname{coker} g \mid \#\operatorname{coker} f$. □

Proposition 4.12. *If $p \mid \kappa_v(G)$, then p divides the determinant of the Cartan matrix of $R(G,T)$.*

Proof. Let Y be the subgroup of $X_*(T)$ generated by the coroots of $R(G,T)$. The order of the cokernel $Y \to \mathrm{Hom}_Z(\mathbb{Z}R(G,T), \mathbb{Z})$ is equal to the determinant of the Cartan matrix. □

Corollary 4.13. *If $p \mid \kappa_v(G)$, then p is not a very good prime for G. Moreover, if G does not contain a normal subgroup of type A_l, then p is a bad prime for G and $p \in \{2,3\}$.*

Proof. By [Hum78, 11.4, Exercise 2] the determinants of the Cartan matrices for the irreducible root systems are:

$$A_l : l+1;\ B_l : 2;\ C_l : 2;\ D_l : 4;\ E_6 : 3;\ E_7 : 2;\ E_8, F_4 \text{ and } G_2 : 1.$$

Compare this with the notion of a prime that is not a very good prime. Then p divides:

$$A_l : l+1;\ B_l : 2;\ C_l : 2;\ D_l : 2;\ E_6 : 2,3;\ E_7 : 2,3;\ E_8 : 2,3,5;\ F_4 : 2,3;\ G_2 : 2,3.$$ □

4.5 Howe's conjecture in bad characteristic

In this section, we show that Howe's conjecture does not hold for $\mathbb{F}$-split groups in bad characteristic. The calculations in the actual group are postponed to the end of this section. Under the assumption that there exists a bad pair, we will construct sets of linearly independent distributions in $J_L(\omega)$ of arbitrary finite size. The support of these distributions is contained in the set of nilpotent elements. Two consequences of our method are the existence of infinitely many regular nilpotent orbits and the inseparability of the regular nilpotent orbit.

4.5.1 Reduction to bad pairs

Let G be an $\mathbb{F}$-split reductive group. Let T be a maximal $\mathbb{F}$-split torus. Let R^+ be a system of positive roots. Let U^+ be the unipotent subgroup corresponding to R^+ and $\mathfrak{n}$ its Lie algebra. Let $B = TU^+$ be the corresponding Borel subgroup. The set of regular nilpotent elements of $\mathfrak{n}$ is denoted by $\mathfrak{n}'$.

Let $H_1, \ldots, H_r$ and E_γ for $\gamma \in R$ be a Chevalley basis for $\mathfrak{g}$. Let $u_\gamma : \mathbb{F} \to U_\gamma$, for $\gamma \in R$, be the corresponding parametrization of U_γ ($\mathrm{d}u_\gamma(1) = E_\gamma$). Now $\mathfrak{n}$ has $E_\alpha : \alpha \in R^+$ as basis. For $x \in \mathfrak{n}$ and $\alpha \in R^+$ we define $x_\alpha \in \mathbb{F}$ such that $x = \sum_{\alpha \in R^+} x_\alpha E_\alpha$. Define $X_\alpha(x) := x_\alpha$.

Definition 4.14. *Let $\eta : \mathbb{F} \to \mathfrak{n}'$ and $\chi : \mathfrak{n}' \to \mathbb{F}$ be polynomial functions. The pair (η, χ) is called a* bad pair *if it satisfies the following four conditions:*

1. *$\chi\eta(\alpha) = \alpha$ for all $\alpha \in \mathbb{F}$.*

2. *If $n, n' \in \mathfrak{n}'$ are conjugate by an element of $G(\mathbb{F})$, then $\chi(n) \equiv \chi(n') \mod \mathbb{F}^{(p)}$. There is $z \in p\mathbb{Z}$ such that $c^z\chi(n) = \chi(cn)$ for all $c \in \mathbb{F}^\times$ and $n \in \mathfrak{n}'$.*

3. *If $\gamma \in R^+$ and $\alpha \in \mathcal{O}^\times$, then $\eta(\alpha)_\gamma \in \mathcal{O}$, and if, in addition, $\gamma \in \Delta$, then $\eta(\alpha)_\gamma \in \mathcal{O}^\times$.*

4. *$\chi \in \mathcal{O}[X_\gamma, X_\beta^{-1} : \gamma \in R^+, \beta \in \Delta] \subset \mathbb{F}[\mathfrak{n}']$, where $\mathbb{F}[\mathfrak{n}']$ is the algebra of all $\mathbb{F}$-regular functions on $\mathfrak{n}'$.*

For the remainder of this subsection, we assume that (η, χ) is a bad pair. Since $\mathbb{F}/\mathbb{F}^{(p)}$ is infinite, the first and second conditions of a bad pair already imply that there are infinitely many regular nilpotent orbits in $\mathfrak{g}$. We will use χ to define G-invariant distributions and η to show that they are linearly independent.

For $n \in \mathbb{N}$ we define a compact subgroup K_n of G as follows (see [MP94] for a general construction):

$$\begin{aligned} U_{\gamma,n} &:= u_\gamma(v^{-1}[n, \infty)), \\ T_{i,n} &:= \{t \in T \mid \forall[\alpha \in X^*(T)]\ v(\alpha(t) - 1) \geq n\}. \end{aligned}$$

Define K_n to be the subgroup of G generated by the groups $U_{\gamma,n}$ and $T_{i,n}$. Define $K := K_0$. Now K is the group of $\mathcal{O}$-points of a split reductive $\mathcal{O}$-group scheme $\mathcal{K}$ with generic fiber G, see [MP94, §3.2].
We may identify $\mathfrak{t}$ with $X_*(T) \otimes_{\mathbb{Z}} \mathbb{F}$ by

$$X_*(T) \ni \gamma \mapsto \mathrm{d}\gamma(1) \in \mathfrak{t}.$$

Let $\delta_1, \ldots, \delta_s$ be a basis for $X_*(T)$ and $H'_1, \ldots, H'_s$ the corresponding basis in $\mathfrak{t}$.
Let L be the $\mathcal{O}$-lattice spanned by $H'_1, \ldots, H'_s$ and all E_γ.
For $m \in \mathfrak{g}$ we define $m_i \in \mathbb{F}$ and $m_\gamma \in \mathbb{F}$ such that

$$m = \sum_{i=1}^{s} m_i H'_i + \sum_{\gamma \in R} m_\gamma E_\gamma.$$

Now L is K-invariant. Thus K acts on $L/\varpi^n L$. The group K_n acts trivially on $L/\varpi^n L$, by the choice of K_n and L. Now $G(\mathbb{F}_q) := \mathcal{K}(\mathbb{F}_q) \cong K/K_1$ and $\mathfrak{g}(\mathbb{F}_q) = L/\varpi L$. Let $B(\mathbb{F}_q) := (K \cap B)/K_1$ and let $\mathfrak{n}(\mathbb{F}_q)$ be its Lie algebra.

Lemma 4.15. *There exists a $N > 0$ such that for all $n \in \mathbb{N}_{>0}$, $k \in K$ and $\alpha \in \mathcal{O}^\times$:*

$$k\eta(\alpha)k^{-1} \in \mathfrak{n} + \varpi^{Nn} L \Rightarrow k \in (B \cap K)K_n.$$

Proof. The map $\pi_0 : G(\mathcal{O}) \to \mathcal{K}(\mathbb{F}_q)$ induces a map on the Lie algebra: $\pi_0 : \mathfrak{g}(\mathcal{O}) \to \mathfrak{g}(\mathbb{F}_q)$ with kernel ϖL. Since $\pi_o(\eta(\alpha))$ is also a regular nilpotent element by condition 3 of Definition 4.14 and $\pi_0(k\eta(\alpha)k^{-1}) \in \mathfrak{n}(\mathbb{F}_q)$, we have $\pi_0(k) \in B(\mathbb{F}_q)$. Thus $k \in (B \cap K)K_1$. Take for the moment a general $N \in \mathbb{N}_{>0}$. Because $\mathfrak{n}$ and $\varpi^{Nn} L$ are $(B \cap K)$-invariant and $K_1 = (B \cap K_1)(U^- \cap K_1)$, we may assume $k \in U^- \cap K$.

Take $x_\gamma \in \mathcal{O}$ arbitrary for $\gamma \in R^-$. Define $u = \prod_{\gamma \in R^-} u_\gamma(x_\gamma)$. Let, for $i = 1, \ldots, s$ and $\beta \in R^-$, $p_i, p_\beta \in \mathbb{F}[X_\gamma : \gamma \in R^-, Y, Y^{-1}]$ be such that $(u\eta(\alpha)u^{-1})_i = p_i(x_\gamma, \alpha, \alpha^{-1})$ and $(u\eta(\alpha)u^{-1})_\beta = p_\beta(x_\gamma, \alpha, \alpha^{-1})$.
Let I be the ideal generated by p_β for $\beta \in R^-$. Then $u\eta(\alpha)u^{-1} \in \mathfrak{n}$ if and only if $p_\beta(x_\gamma, \alpha, \alpha^{-1}) = 0$ for all $\beta \in R^-$. Because of Corollary 4.5 for $x_\gamma, \alpha \in \overline{\mathbb{F}}$:

$$u\eta(\alpha)u^{-1} \in \mathfrak{n} \Leftrightarrow u = 1 \Leftrightarrow \forall \gamma \in R^- \ [x_\gamma = 0],$$

where $u = \prod_{\gamma \in R^-} u_\gamma(x_\gamma)$. By the Nullstellensatz we have $X_\gamma \in \sqrt{I}$ for all $\gamma \in R^-$. Thus there exists $m \in \mathbb{N}$ such that $X_\gamma^m \in I$ for all $\gamma \in R^-$. Therefore, there are polynomials $f_{\gamma,\beta} \in \mathbb{F}[X_\gamma, Y, Y^{-1}]$ such that

$$X_\gamma^m = \sum_{\beta \in R^+} f_{\gamma,\beta} p_\beta.$$

Let M be the smallest $n \in \mathbb{N}_{\geq 0}$ such that $f_{\gamma,\beta}(x_\gamma, \alpha, \alpha^{-1}) \subset \varpi^{-n}\mathcal{O}$ for all $\beta, \gamma \in R^-$, $x_\gamma \in \mathcal{O}$ and $\alpha \in \mathcal{O}^\times$.
Take $N := m + M$. Assume $u\eta(\alpha)u^{-1} \in \mathfrak{n} + \varpi^{Nn}L$, then $v(p_\beta(x_\gamma, \alpha, \alpha^{-1})) \geq Nn$. Since

$$x_\gamma^m = \sum_{\beta \in R^-} f_{\gamma,\beta}(x_\gamma, \alpha, \alpha^{-1}) p_\beta(x_\gamma, \alpha),$$

we have

$$v(x_\gamma^m) \geq Nn - M = mn + (n-1)M \geq mn.$$

Thus $v(x_\gamma) \geq n$. Hence if $k\eta(\alpha)k^{-1} \in \mathfrak{n} + \varpi^{nN}L$ and $k \in U^- \cap K$, then $k \in K_n$. $\square$

Let δ_B be the modular function of B, thus

$$\delta_B(b) \int_B f(xb)\mathrm{d}x = \int_B f(x)\mathrm{d}x.$$

Proposition 4.16 (Ranga Rao). *Assume that $V \subset \mathfrak{n}$ is open and B-invariant. Then for all $f \in C_c^\infty(\mathfrak{g})$,*

$$\int_V f(bXb^{-1})\mathrm{d}X = \delta_B(b) \int_V f(X)\mathrm{d}X.$$

Moreover, the distribution

$$D_V(f) := \int_V \int_K f(kXk^{-1})\mathrm{d}k\mathrm{d}X$$

is G-invariant.

Proof. Since $\delta|_B(b) = |\det(\mathrm{Ad}\ b|_{\mathfrak{n}})|^{-1}$,

$$\int_{\mathfrak{n}} f(bXb^{-1})\mathrm{d}X = \delta_B(b) \int_{\mathfrak{n}} f(X)\mathrm{d}X.$$

We can apply this formula to $\int_V f(X)\mathrm{d}X$, because V is open and B-invariant. This proves the first statement of the Proposition.
The second statement follows from the first by [How74, Proposition 4]. The method described here is essentially in [Rao72]. $\square$

Let $\mathfrak{n}(\mathcal{O})$ be the $\mathcal{O}$-module generated by E_α, with $\alpha \in R^+$.

Corollary 4.17. *Let $\omega \subset \mathfrak{g}$ be open and compact. If $V \subset \mathfrak{n}$ is open and B-invariant, then $D_V \in J(\omega)$.*

Proof. By Proposition 4.16, D_V is a G-invariant distribution. The support of D_V is contained in $\overline{\mathfrak{n}^K}$. Since ω is open, there is $m \in \mathbb{N}$ such that $\varpi^m \mathfrak{n}(\mathcal{O}) \subset \omega$. Since $\mathfrak{n}(\mathcal{O})^T = \mathfrak{n}$, then supp $D_V \subset \overline{\mathfrak{n}^K} \subset \overline{(\omega^T)^K} \subset \overline{\omega^G}$. $\square$

For $\alpha \in \mathbb{F}^\times$ and $s \in \mathbb{N}$, define $V_{\alpha,s} \subset \mathfrak{n}$ as follows:

$$V_{\alpha,s} := \{n \in \mathfrak{n} \mid \chi(n) \equiv \alpha \mod (\varpi^s \mathcal{O} + \mathbb{F}^{(p)})\}.$$

Let $\Delta = \{\alpha_1, \ldots, \alpha_m\}$. For $a_1, \ldots, a_m \in \mathbb{F}$, define the following nilpotent element:

$$n(a_1, \ldots, a_m) := \sum_{i=1}^m a_i E_{\alpha_i}.$$

Take $z \in p\mathbb{Z}$ such that $\chi(cn) = c^z \chi(n)$ for all $n \in \mathfrak{n}'$ and $c \in \mathbb{F}$.

Lemma 4.18. *Let $N \in \mathbb{N}$ be the constant arising from Lemma 4.15. Let $n \in \mathbb{N}_{>0}$, $\alpha \in \mathcal{O}^\times$ and $\beta \in \mathcal{O}^\times$.*
If

$$\int_{V_{\varpi^{-znN}\beta,n}} \int_{k\in K} 1_{\varpi^{-Nn}\eta(\alpha)+L}(kXk^{-1}) \mathrm{d}k \mathrm{d}X > 0,$$

then $\alpha \equiv \beta \mod \varpi^n \mathcal{O} + \mathcal{O}^{(p)}$.

Proof. Let $X \in V_{\varpi^{-znN}\beta,n}$, $k' \in K$ and $l' \in L$ such that

$$k'Xk'^{-1} + l' = \varpi^{-Nn}\eta(\alpha).$$

Since L is K-invariant, there exist $k \in K$ and $l \in L$ such that

$$k\varpi^{-nN}\eta(\alpha)k^{-1} + l = X \in V_{\varpi^{n-znN}\beta,n} \subset \mathfrak{n}.$$

Thus $k \in (K \cap B)K_n$ by Lemma 4.15, because $k\eta(\alpha)k^{-1} \in \mathfrak{n} + \varpi^{nN}L$. Take $b_k \in K \cap B$ and $k_n \in K_n$ such that $k = k_n b_k$. Take $a_1, \ldots, a_m \in \mathcal{O}^\times$ and $n_2 \in \mathfrak{n}_2(\mathcal{O})$ such that $b_k\eta(\alpha)b_k^{-1} = n(a_1, \ldots, a_m) + n_2$. By condition 2 of Definition 4.14, there exists $\gamma \in \mathbb{F}$ such that $\chi(n(a_1, \ldots, a_m) + n_2) = \alpha + \gamma^p$. Since $k_n \in K_n$ and $n(a_1, \ldots, a_m) + n_2 \in L$, there exists $l' \in L$ such that
$k_n(n(a_1, \ldots, a_m) + n_2)k_n^{-1} = n(a_1, \ldots, a_m) + n_2 + \varpi^n l'$. Thus

$$\begin{aligned}\chi(k\eta(\alpha)k^{-1} + \varpi^{nN}l) &= \chi(k_n b_k \eta(\alpha) b_k^{-1} k_n^{-1} + \varpi^{nN}l) \\ &= \chi(n(a_1, \ldots, a_m) + n_2 + \varpi^n l' + \varpi^{nN}l).\end{aligned}$$

Since the a_i are in $\mathcal{O}^\times$ and $\chi \in \mathcal{O}[X_\gamma, X_\beta^{-1} : \gamma \in R^+, \beta \in \Delta]$,

$$\begin{aligned}\chi(n(a_1, \ldots, a_m) + n_2 + \varpi^n l' + \varpi^{nN}l) &\equiv \chi(n(a_1, \ldots, a_m) + n_2) \\ &= \alpha + \gamma^p \mod \varpi^n \mathcal{O}.\end{aligned}$$

Since $\chi(\varpi^{-nN}x) = \varpi^{-znN}\chi(x)$ for all $x \in \mathfrak{g}$,

$$\chi(k\varpi^{-nN}\eta(\alpha)k^{-1} + l) \equiv (\alpha + \gamma^p)\varpi^{-znN} \mod \varpi^{n-znN}\mathcal{O}.$$

Since $k\varpi^{-nN}\eta(\alpha)k^{-1} + l \in V_{\varpi^{-znN}\beta,n}$,

$$\chi(k\varpi^{-nN}\eta(\alpha)k^{-1} + l) \equiv \varpi^{-znN}\beta \mod (\varpi^{n-znN}\mathcal{O} + \mathbb{F}^{(p)}).$$

Thus

$$\varpi^{-znN}\beta \equiv \chi(k\varpi^{-nN}\eta(\alpha)k^{-1} + l) \equiv \varpi^{-znN}\alpha \mod (\varpi^{n-znN}\mathcal{O} + \mathbb{F}^{(p)}).$$

Then $\alpha \equiv \beta \mod (\varpi^n\mathcal{O} + \mathbb{F}^{(p)})$. Since $\mathbb{F}^{(p)} \cap \mathcal{O} = \mathcal{O}^{(p)}$ and $\alpha, \beta \in \mathcal{O}$, the lemma follows. □

Theorem 4.19. *Let G be an $\mathbb{F}$-split reductive group. If there exists a bad pair (η, χ) for G, then* $\dim J_L(\omega) = \infty$.

Proof. Take $n \in \mathbb{N}_{>0}$. Let $\alpha_1, \ldots, \alpha_k$ be representatives of the cosets of $\varpi^n\mathcal{O} + \mathcal{O}^{(p)}$ in $\mathcal{O}$. For $1 \le i \le k$, define the following distribution and function:

$$\begin{aligned} D_i(f) &:= D_{V_{\varpi^{-znN}\alpha_i,n}}(f) = \int_{V_{\varpi^{-znN}\alpha_i,n}} \int_{k\in K} f(kXk^{-1})\mathrm{d}k\mathrm{d}X, \\ f_i &:= 1_{\varpi^{-nN}\eta(\alpha_i)+L}. \end{aligned}$$

The distributions D_i are in $J(\omega)$ by Corollary 4.17. Let $c_i := D_i(f_i)$, then $c_i > 0$. Then $D_i(f_j) = c_i\delta_{ij}$ by Lemma 4.18 . Therefore, the distributions $D_1, \ldots, D_k$ are linearly independent. Since we may view the f_i's as elements of $C_c^\infty(\mathfrak{g}/L)$, the distributions D_i remain linearly independent when we view them as distributions of $\mathfrak{g}/L$. Thus $\dim J_L(\omega) \ge k$. As n goes to infinity so does k. □

4.5.2 The bad pair construction

In this subsection we assume that $p = \operatorname{char} \mathbb{F}$ is bad for G. The construction of a bad pair is done in three steps. First we construct a bad pair in the case where G is simple of adjoint type. Then we show that if there is a bad pair for $\mathrm{Ad}(G)$, then we can construct a bad pair for G. In the third step, we combine the results of the first and second step to construct a bad pair.

Define $X := \sum_{\beta\in\Delta} E_\beta$. Let $\alpha_1, \ldots, \alpha_k$ be the roots of height $p+1$. They exist because p is bad for G. Define

$$n(a_1, \ldots, a_k) := X + \sum_{i=1}^{k} a_i E_{\alpha_i},$$

for $a_i \in \mathbb{F}$. Recall that $\mathfrak{n}_i = \bigoplus_{\alpha:\mathrm{ht}(\alpha)=i} \mathfrak{g}_\alpha$.

Theorem 4.20 (Springer)**.** *Let G be a simple group. Let $X := \sum_{\alpha\in\Delta} E_\alpha$. Assume that $p = \operatorname{char} \mathbb{F}$ is bad for G. Then*

1. $[X,\cdot]:\mathfrak{n}_i\to\mathfrak{n}_{i+1}$ *is surjective for* $1\le i\le p-1$.

2. $\dim\mathfrak{n}_{p+1}/[X,\mathfrak{n}_p]=1$.

3. $|\Delta|=\dim\mathfrak{n}_1=\dim\mathfrak{n}_i+1$ *for* $2\le i\le p+1$.

In [Spr66] the operation $[X,\cdot]:\mathfrak{g}\to\mathfrak{g}$ is studied in more detail, see for example [Spr66, (2.6) Theorem.].

Proof. See [Spr66, (2.11) Theorem.]. □

Corollary 4.21. *If p is bad for G, then the regular nilpotent orbit is not separable.*

Proof. Since X is a regular nilpotent element,

$$\dim Z_G(X)=\dim Z_B(X)=\dim B-\dim U=\dim T.$$

Since $\dim Z_T(X)=\dim T-|\Delta|$, we have $\dim Z_U(X)=|\Delta|$.
Thus if the orbit of X is separable, then $[X,\cdot]:\mathfrak{n}\mapsto\mathfrak{n}_{\ge 2}$ has a kernel of dimension $|\Delta|$. Therefore, $[X,\cdot]$ must be surjective. As Theorem 4.20 shows, this is not the case when p is bad for a simple group G. By passing to the adjoint group, the corollary follows. □

Proposition 4.22. *Assume that G is a simple group. There exists a surjective linear function $f:\mathbb{F}^k\to\mathbb{F}$ such that if $n(a_1,\dots,a_k)$ is conjugate to $n(b_1,\dots,b_k)$, then $f(a_1,\dots,a_k)\equiv f(b_1,\dots,b_k)\mod\mathbb{F}^{(p)}$.*

Proof. Let $f:\mathfrak{n}_{p+1}\to\mathbb{F}$ be a linear function corresponding to the isomorphism $\mathfrak{n}_{p+1}/[X,\mathfrak{n}_p]\cong\mathbb{F}$. For $u\in U$ write $u=\prod_{\gamma\in R^+}u_\gamma(x_\gamma)$.
By Theorem 4.20, we have $\dim\mathfrak{n}_i=\dim\mathfrak{n}_1-1$ for $2\le i\le p+1$, and $n\mapsto[X,n]$ is a bijection from $\mathfrak{n}_i$ to $\mathfrak{n}_{i+1}$ for $2\le i\le p-1$.

We will prove by induction on the height of the roots that there exist $c_\gamma,d_\gamma\in\mathbb{F}$ and $x\in\mathbb{F}$, such that for $i\le p-1$, if $uXu^{-1}\equiv X\mod\mathfrak{n}_{\ge i+2}$, then

$$x_\gamma=c_\gamma x^{\mathrm{ht}(\gamma)},\tag{4.1}$$

for $\gamma\in R^+$ with $\mathrm{ht}(\gamma)\le i-1$ and

$$uXu^{-1}\equiv X-\left[X,\sum_{\gamma\in R^+_{i+1}}x_\gamma E_\gamma\right]+\sum_{\gamma\in R^+_{i+2}}d_\gamma x^{i+1}E_\gamma\mod\mathfrak{n}_{\ge i+3}.\tag{4.2}$$

Before we give the induction argument, first we restate (4.2).
The nilpotent element $uXu^{-1}\mod\mathfrak{n}_{\ge i+3}$ only depends on the value of x_γ for these γ with height at most $i+1$. In expression (4.2) the dependence on the roots of height $i+1$ is taken care of with the term $-[X,\sum_{\gamma\in R^+_{i+1}}x_\gamma E_\gamma]$. So for the proof of (4.2) we need to show that

$$RM_i:=uXu^{-1}-X+[X,\sum_{\gamma\in R^+_{i+1}}x_\gamma E_\gamma]$$

is equal to $\sum_{\gamma \in R^+_{i+2}} d_\gamma x^{i+1} E_\gamma$, when $uXu^{-1} \equiv X \mod \mathfrak{n}_{\geq i+2}$.

The function $[X, \cdot] : \mathfrak{n}_1 \to \mathfrak{n}_2$ gives that $x := x_\gamma = x_\delta$ for all $\gamma, \delta \in \Delta$. By the Steinberg conjugacy formula [Spr98, Proposition 8.2.3] we have $d_\gamma \in \mathbb{F}$ such that $RM_1 = \sum_{\gamma \in R^+_3} d_\gamma x^2 E_\gamma$.
Assume that (4.1) and (4.2) hold for $i - 1$.
The function $[X, \cdot] : \mathfrak{n}_i \to \mathfrak{n}_{i+1}$ is bijective. So for all $n_{i+1} \in \mathfrak{n}_{i+1}$ there is exactly one $u \in U_i$ such that $u(X + n_{i+1})u^{-1} = X \mod \mathfrak{n}_{\geq i+2}$, namely, the one corresponding to the inverse of $[X, \cdot]$. Let $\mathrm{In} : \mathfrak{n}_{i+1} \to \mathfrak{n}_i$ be the inverse of $[X, \cdot]$. Then the $x_\gamma \in \mathbb{F}$ for $\gamma \in R^+_i$ are such that

$$\mathrm{In}(RM_{i-1}) = \sum_{\gamma \in R^+_i} x_\gamma E_\gamma.$$

By the induction hypothesis, $RM_{i-1} = \sum_{\gamma \in R^+_{i+1}} d_\gamma x^i E_\gamma$ for some constants $d_\gamma \in \mathbb{F}$. Thus for every $\gamma \in R^+_i$ there exists $c_\gamma \in \mathbb{F}$ such that $x_\gamma = c_\gamma x^i$. By the Steinberg conjugacy formula, we have $d_\gamma \in \mathbb{F}$ such that $RM_i = \sum_{\gamma \in R^+_{i+2}} d_\gamma x^{i+1} E_\gamma$.

Assume that $un(a_1, \ldots, a_k)u^{-1} \equiv n(b_1, \ldots, b_k) \mod \mathfrak{n}_{\geq p+2}$.
Then certainly

$$uXu^{-1} \equiv un(a_1, \ldots, a_k)u^{-1} \equiv n(b_1, \ldots, b_p) \equiv X \mod \mathfrak{n}_{\geq p+1}.$$

Thus by (4.2)

$$un(a_1, \ldots, a_k)u^{-1} \equiv [X, n_p] + n(a_1 + d_1 x^p, \ldots, a_k + d_k x^p) \mod \mathfrak{n}_{\geq p+2},$$

with $d_i := d_{\alpha_i}$ and $n_p \in \mathfrak{n}_p$. Thus

$$n(a_1 + d_1 x^p, \ldots, a_k + d_k x^p) - X \equiv n(b_1, \ldots, b_k) - X \mod [X, \mathfrak{n}_p].$$

Since $f : \mathfrak{n}_{p+1} \to \mathbb{F}$ is a linear map with kernel $[X, \mathfrak{n}_p]$,

$$f(b_1 - a_1, \ldots, b_k - a_k) = f(d_1 x^p, \ldots, d_k x^p) = x^p f(d_1, \ldots, d_k).$$

Define $c := f(d_1, \ldots, d_k)$. Then $f(b_1 - a_1, \ldots, b_k - a_k) \in c\mathbb{F}^{(p)}$ if and only if $n(a_1, \ldots, a_k)$ is U-conjugate modulo $\mathfrak{n}_{\geq p+2}$ to $n(b_1, \ldots, b_k)$.
Since being U-conjugate modulo $\mathfrak{n}_{\geq p+2}$ is an equivalence relation, we have that $c \in \mathbb{F}^{(p)}$.
Now $c \neq 0$, because by Proposition 4.4 over an algebraically closed field the orbit of X in $\mathfrak{n}$ contains $X + \bigoplus_{i=2}^{\mathrm{ht}(R)} \mathfrak{n}_i$. Thus $c \in (\mathbb{F}^\times)^p$. □

Lemma 4.23. *If G is simple of adjoint type and p is bad for G, then there exists a bad pair for G.*

Proof. Choose an $\alpha \in \Delta$, define

$$U_{\hat{\alpha}} := \prod_{\gamma \in \{\gamma \in R^+ - \{\alpha\} | \mathrm{ht}(\gamma) \leq p-1\}} U_\gamma.$$

Recall that $X = \sum_{\alpha \in \Delta} E_\alpha$. The map $\Phi : X_*(T) \to \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}R(G,T), \mathbb{Z})$ is an isomorphism because G is of adjoint type. For $\alpha \in \Delta$ take cocharacters $\omega_\alpha \in X_*(T)$ such that for all $\beta \in \Delta$, $\langle \omega_\alpha, \beta \rangle = \delta_{\alpha,\beta}$.

Let $n = \sum_{\alpha\in\Delta} c_\alpha E_\alpha$. Take $t = \prod_{\alpha\in\Delta} \omega_\alpha(c_\alpha)$, then $tXt^{-1} = n$. Let $s \in T$. Assume that $sXs^{-1} = n$, then for all $\alpha \in \Delta$, $c_\alpha = \alpha(s)$. Let $d_\alpha \in \mathbb{F}$ be such that $s = \prod_{\alpha\in\Delta} X_\alpha(d_\alpha)$. Since the X_α are a basis for $X_*(T)$ such d_α's exist. Thus $d_\alpha = \alpha(t) = c_\alpha$. Hence $s = \prod_{\alpha\in\Delta} \omega_\alpha(c_\alpha) = t$.

Thus for every $n \in \mathfrak{n}'$ there is exactly one $t \in T$ such that $tnt^{-1} \in X + \mathfrak{n}_{\geq 2}$. Therefore, by the proof of Proposition 4.22, for every $n \in \mathfrak{n}'$, there exists a unique $b \in TU_{\hat{\alpha}}$ such that $bnb^{-1} = X + n(a_1, \ldots, a_k) + n_{p+2}$, with $n_{p+2} \in \mathfrak{n}_{\geq p+2}$. Write $n = \sum_{\alpha\in R} x_\alpha E_\alpha$. The $a_1, \ldots, a_k$ depend on x_α for $\alpha \in R_i^+$ with $i \leq p+1$. Let f_i be the rational functions such that $a_i = f_i(x_\alpha)$. The f_i are homogeneous of degree $-p$:

$$\begin{aligned}
bnb^{-1} &\equiv X + n(a_1, \ldots, a_k) && \mod \mathfrak{n}_{\geq p+2}, \\
b\lambda nb^{-1} &\equiv \lambda(X + n(a_1, \ldots, a_k)) && \mod \mathfrak{n}_{\geq p+2}, \\
tb\lambda nb^{-1}t^{-1} &\equiv X + \frac{\lambda}{\lambda^{p+1}} n(a_1, \ldots, a_k) && \mod \mathfrak{n}_{\geq p+2},
\end{aligned}$$

where $t \in T$ is such that $\gamma(t) = \frac{1}{\lambda}$ for all $\gamma \in \Delta$.
Define $\chi(n) := f(f_1(x_\alpha), \ldots, f_k(x_\alpha))$ for $n \in \mathfrak{n}$.
Choose a $g : \mathbb{F} \to \mathbb{F}^k$ to be a right inverse of f, that is, $fg = \mathrm{id}$, such that $n(g(\mathcal{O})) \subset \mathfrak{n}_{p+1}(\mathcal{O})$. Define $\eta : \mathbb{F} \to \mathfrak{n}'$ by $\eta(a) := X + n(g(a))$. Now (χ, η) is a bad pair for G. $\square$

Now we will deduce from Lemma 4.23 the existence of a bad pair for reductive groups G for which p is bad.

Lemma 4.24. *If (η, χ) is a bad pair for $\mathrm{Ad}(G)$, then there exists a bad pair for G.*

Proof. Let $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ be the natural morphism. The linear map $\mathrm{d}(\mathrm{Ad}) : \mathfrak{n} \to \mathfrak{n}^{\mathrm{ad}}$ is a bijection by Lemma 4.9. Let $\mathrm{d}a : \mathfrak{n}^{\mathrm{ad}} \to \mathfrak{n}$ be its inverse. If $n, n' \in \mathfrak{n}$ are conjugate by G, then their image is conjugate by $\mathrm{Ad}(G)$. If E_α is a Chevalley basis for G, then $\mathrm{d}(\mathrm{Ad})(E_\alpha)$ is one for $\mathrm{Ad}(G)$. Thus $(\mathrm{d}a \circ \eta, \chi \circ \mathrm{d}(\mathrm{Ad}))$ is a bad pair for G. $\square$

Theorem 4.25. *Assume that G is an $\mathbb{F}$-split reductive group and p is bad for G. Then there exists a bad pair for G.*

Proof. By Lemma 4.24 we may assume that G is semisimple of adjoint type. Then G is a direct product of simple connected normal subgroups. Since the characteristic is bad, there exists a bad pair for at least one of these subgroups by Lemma 4.23. Therefore, there exists a bad pair for G. $\square$

Theorem 4.26. *If G is an $\mathbb{F}$-split reductive group and char $\mathbb{F}$ is bad, then there are infinitely many nilpotent orbits and Howe's conjecture on the Lie algebra does not hold.*

Proof. This follows from Theorem 4.25 and Theorem 4.19. $\square$

4.5.3 The example $SO_5(\mathbb{F})$, char $\mathbb{F} = 2$

In this subsection $\mathbb{F}$ has characteristic 2. We follow [Spr98, §7.4.7(6)] for the definition of $SO_5(\mathbb{F})$. Let $V = \mathbb{F}^5$ and let Q be the quadratic form on V defined by

$$Q(e_0, e_1, e_2, e_3, e_4) := e_0^2 + e_1e_3 + e_2e_4.$$

Now we define $SO_5(\mathbb{F})$ to be the subgroup of $t \in GL(V)$ with $Q(tv) = Q(v)$ for all $v \in V$. Then

$$T := \{t(t_1, t_2) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 & 0 \\ 0 & 0 & t_2 & 0 & 0 \\ 0 & 0 & 0 & t_1^{-1} & 0 \\ 0 & 0 & 0 & 0 & t_2^{-1} \end{pmatrix} : t_i \in \mathbb{F}^\times\}$$

is a maximal torus of SO_5 that is $\mathbb{F}$-split. Define, for $i = 1, 2$, the character ϵ_i of T by

$$\epsilon_i(t(t_1, t_2)) := t_i.$$

Then $R(G, T) = \{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j \mid i \neq j\}$. Let $R^+ := \{\epsilon_1 - \epsilon_2, \epsilon_2, \epsilon_1, \epsilon_1 + \epsilon_2\}$ be a system of positive roots and $\Delta := \{\epsilon_1 - \epsilon_2, \epsilon_2\}$ the corresponding set of simple roots. We take the following basis for $\mathfrak{n}$:

$$E_{\epsilon_1-\epsilon_2} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad E_{\epsilon_1} := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{\epsilon_2} := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad E_{\epsilon_1+\epsilon_2} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now $E_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in R^+$.
Thus $X := E_{\epsilon_1-\epsilon_2} + E_{\epsilon_2}$. Also $\mathfrak{n}_2 = \mathfrak{g}_{\epsilon_1}$ and $\mathfrak{n}_3 = \mathfrak{g}_{\epsilon_1+\epsilon_2}$. The linear map $[X, \cdot] : \mathfrak{n}_2 \to \mathfrak{n}_3$ is 0, since both $E_{\epsilon_1-\epsilon_2}$ and E_{ϵ_2} commute with E_{ϵ_1}. Thus according to Proposition 4.22 and its proof, for $d, d' \in \mathbb{F}$, $X + dE_{\epsilon_1+\epsilon_2}$ is U-conjugate to $X + d'E_{\epsilon_1+\epsilon_2}$ if and only if $d \equiv d'$ mod $\mathbb{F}^{(2)}$. Now we follow Lemma 4.23. We take $U_{\hat{\epsilon}_2} := U_{\epsilon_1-\epsilon_2}$. Define for $a, b, c, d \in \mathbb{F}$

$$n(a, b, c, d) := aE_{\epsilon_1-\epsilon_2} + bE_{\epsilon_2} + cE_{\epsilon_1} + dE_{\epsilon_1+\epsilon_2}.$$

Assume that $a, b \neq 0$, then by the proof of Lemma 4.23 there is a unique $g \in TU_{\hat{\epsilon}_2}$ such that $gn(a, b, c, d)g^{-1} = X + d'E_{\epsilon_1+\epsilon_2}$ for some $d' \in \mathbb{F}$. We compute d': first get the a and b to 1 by conjugating with $t := t((ab)^{-1}, b^{-1})$, then

$$tn(a, b, c, d)t^{-1} = n(1, 1, c(ab)^{-1}, da^{-1}b^{-2}).$$

By conjugating the result with

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & c(ab)^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c(ab)^{-1} & 1 \end{pmatrix}$$

we get $n(1,1,0,da^{-1}b^{-2})$. Thus $d' = \frac{d}{ab^2}$. Assume that $a', b' \neq 0$. Thus $n(a,b,c,d)$ is conjugate to $n(a',b',c',d')$ if and only if $\frac{d}{ab^2} \equiv \frac{d'}{a'b'^2} \mod \mathbb{F}^{(2)}$. So we define $\chi : \mathfrak{n}' \to \mathbb{F}$ by $\chi(n(a,b,c,d)) := \frac{d}{ab^2}$ and we define $\eta : \mathbb{F} \to \mathfrak{n}'$ by $\eta(d) := n(1,1,0,d) = E_{\epsilon_1-\epsilon_2} + E_{\epsilon_2} + dE_{\epsilon_1+\epsilon_2}$. Then (η, χ) is a bad pair for SO_5.

4.6 Howe's conjecture and $\kappa_v(G)$

In this section we assume that p divides $\kappa_v(G)$, i.e., the characteristic of $\mathbb{F}$ divides the order of the cokernel of the map

$$\Phi : X_*(T) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R(G,T), \mathbb{Z}),$$
$$\gamma \mapsto (\alpha \mapsto \langle \gamma, \alpha \rangle).$$

We will follow the same strategy as in Section 4.5.1. By the proof of Proposition 4.6 there are integers $z_i \in \mathbb{Z}$ such that $\kappa : \mathfrak{n}' \to \mathbb{F}^\times$ defined by $\kappa(n) := \prod_{i=1}^m n_{\alpha_i}^{z_i}$ is surjective and $\kappa : \mathfrak{n}' \to \mathbb{F}^\times/(\mathbb{F}^\times)^p$ is B-invariant. Take $\nu : \mathbb{F}^\times \to \mathfrak{n}$ such that ν is algebraic and $\kappa\nu$ is the identity. By the proof of Proposition 4.6, we can choose ν in such a way that for all $\alpha \in \mathcal{O}^\times$: $\nu(\alpha)_\gamma \in \mathcal{O}^\times$ for all $\gamma \in \Delta$ and $\nu(\alpha)_\gamma = 0$ for all $\gamma \in R - \Delta$. The pair (ν, κ) plays a similar role in this case as the bad pair (η, χ) in the bad characteristic case.

Let $N > 0$ such that for all $n \in \mathbb{N}_{>0}$, $k \in K$ and $\alpha \in \mathcal{O}^\times$:

$$k\nu(\alpha)k^{-1} \in \mathfrak{n} + \varpi^{Nn}L \Rightarrow k \in (B \cap K)K_n.$$

By Lemma 4.15 such an N exists.

Define the following B-invariant open set of $\mathfrak{n}$:

$$V_{\alpha,s} := \left\{ n \in \mathfrak{n}' \mid \kappa(n) \equiv \alpha \mod (1 + \varpi^s\mathcal{O})(\mathbb{F}^\times)^p \right\}.$$

Define $n(a_1, \ldots, a_m) := \sum_{i=1}^m a_i E_{\alpha_i}$.
Define $z := \sum_{i=1}^m z_i$. Then $\kappa(\varpi^n x) = \varpi^{zn}\kappa(x)$, for all $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$.

Lemma 4.27. *Let $n \in \mathbb{N}_{>0}$, $\alpha \in \mathcal{O}^\times$ and $\beta \in \mathcal{O}^\times$.*
If

$$\int_{V_{\varpi^{-znN}\beta,n}} \int_{k \in K} 1_{\varpi^{-Nn}\nu(\alpha)+L}(kXk^{-1}) \mathrm{d}k \mathrm{d}X > 0,$$

then $\alpha \equiv \beta \mod (1 + \varpi^n\mathcal{O})(\mathcal{O}^\times)^p$.

Proof. Since the integral is positive, there exist a $k \in K$ and $l \in L$ such that

$$k\varpi^{-nN}\nu(\alpha)k^{-1} + l \in V_{\varpi^{-znN}\beta,n} + \mathfrak{n}_{\geq 2} \subset \mathfrak{n}.$$

Since $k\nu(\alpha)k^{-1} \in \mathfrak{n} + \varpi^{nN}L$, we have $k \in (K \cap B)K_n$ by Lemma 4.15. Take $b_k \in K \cap B$ and $k_n \in K_n$ such that $k = k_n b_k$. Take $a_1, \dots, a_m \in \mathcal{O}^\times$ and $n_2 \in \mathfrak{n}_{\geq 2}(\mathcal{O})$ such that $b_k\nu(\alpha)b_k^{-1} = n(a_1, \dots, a_m) + n_2$. By the construction of κ, there exists $\gamma \in \mathbb{F}^\times$ such that $\kappa(n(a_1, \dots, a_m)) = \alpha\gamma^p$. Since $k_n \in K_n$ and $n(a_1, \dots, a_m) + n_2 \in L$, there exists $l' \in L$ such that
$k_n(n(a_1, \dots, a_m) + n_2)k_n^{-1} = n(a_1, \dots, a_m) + n_2 + \varpi^n l'$. Thus

$$\begin{aligned}\kappa(k\nu(\alpha)k^{-1} + \varpi^{nN}l) &= \kappa(k_n b_k \nu(\alpha)b_k^{-1}k_n^{-1} + \varpi^{nN}l) \\ &= \kappa(n(a_1, \dots, a_m) + n_2 + \varpi^n l' + \varpi^{nN}l) = \prod_{i=1}^m (a_i + \varpi^n l_i)^{z_i}\end{aligned}$$

for some $l_i \in \mathcal{O}$. Since the a_i are in $\mathcal{O}^\times$,

$$\prod_{i=1}^m (a_i + \varpi^n l_i)^{z_i} \equiv \prod_{i=1}^m a_i^{z_i} = \alpha\gamma^p \mod (1 + \varpi^n\mathcal{O}).$$

Since $\kappa(\varpi^{-nN}x) = \varpi^{-znN}\kappa(x)$ for all $x \in \mathfrak{g}$,

$$\kappa(k\varpi^{-nN}\nu(\alpha)k^{-1} + l) \equiv (\alpha\gamma^p)\varpi^{-znN} \mod (1 + \varpi^n\mathcal{O}).$$

Since $k\varpi^{-nN}\nu(\alpha)k^{-1} + l \in V_{\varpi^{-znN}\beta,n} + \mathfrak{n}_{\geq 2}$,

$$\kappa(k\varpi^{nN}\nu(\alpha)k^{-1} + l) \equiv \varpi^{-znN}\beta \mod (\mathbb{F}^\times)^p(1 + \varpi^n\mathcal{O}).$$

Thus

$$\varpi^{-znN}\beta \equiv \kappa(k\varpi^{-nN}\nu(\alpha)k^{-1} + l) \equiv \varpi^{-znN}\alpha \mod ((\mathbb{F}^\times)^p(1 + \varpi^n\mathcal{O}).$$

Then $\alpha \equiv \beta \mod (\mathbb{F}^\times)^p(1+\varpi^n\mathcal{O})$. Since $(\mathbb{F}^\times)^p \cap \mathcal{O}^\times = (\mathcal{O}^\times)^p$ and $\alpha, \beta \in \mathcal{O}^\times$, the lemma follows. □

Theorem 4.28. *Let G be an $\mathbb{F}$-split reductive group. Assume that* $\operatorname{char} \mathbb{F} \mid \kappa_v(G)$. *Then Howe's conjecture does not hold.*

Proof. The proof is simular to the one of Theorem 4.19.
Let $\alpha_1, \dots, \alpha_k$ be representatives of the cosets of $(1 + \varpi^n\mathcal{O})\mathcal{O}^\times$ in $\mathcal{O}^\times$. For $1 \leq i \leq k$ define the following distribution and function:

$$\begin{aligned} D_i(f) &:= D_{V_{\varpi^{-znN}\alpha_i,n}}(f), \\ f_i &:= 1_{\varpi^{-nN}\nu(\alpha_i)+L}. \end{aligned}$$

Let $c_i := D_i(f_i) > 0$. By Lemma 4.27, $D_i(f_j) = c_i\delta_{ij}$. Therefore, $\dim J_L(\omega) \geq k$. □

4.7 Howe's conjecture in good characteristic

Howe's conjecture does not hold when the characteristic is bad or $p \mid \kappa_v(G)$. In this section we investigate Howe's conjecture in good characteristic. Throughout this section we assume that p is good for G. The proofs of Howe's conjecture are based on [HC99, Part II].

4.7.1 Associated cocharacters to nilpotent elements

In this subsection we recall the theory of associated cocharacters. Let $\tau \in X_*(G)$. For $z \in \mathbb{Z}$, we define the following subspaces of $\mathfrak{g}$:

$$\begin{aligned}
\mathfrak{g}(z;\tau) &:= \{X \in \mathfrak{g} \mid \forall[a \in \mathbb{F}]\ \tau(a)X\tau(a)^{-1} = a^z X\},\\
\mathfrak{g}(\geq z;\tau) &:= \bigoplus_{i \geq z} \mathfrak{g}(i;\tau),\\
\mathfrak{p}(\tau) &:= \mathfrak{g}(\geq 0;\tau),\\
\mathfrak{n}(\tau) &:= \mathfrak{g}(\geq 1;\tau).
\end{aligned}$$

We sometimes abbreviate $\mathfrak{g}(z;\tau)$ ($\mathfrak{g}(\geq z;\tau)$) by $\mathfrak{g}(z)$ ($\mathfrak{g}(\geq z)$ resp.), in which case the cocharacter τ should be clear from the context.
Let $X \in \mathfrak{g}$ be nilpotent. It is called *distinguished* in $\mathfrak{g}$ if each torus contained in $Z_G(X)$ is contained in the center of G.
A cocharacter τ of G is called *associated* to X if $X \in \mathfrak{g}(2,\tau)$ and if there exists a Levi subgroup L in G such that X is distinguished nilpotent in $\mathfrak{l}$ and such that $\operatorname{im}\tau \subset (L,L)$.
Following [McN04] we define $N(X) := \{g \in G \mid \mathrm{Ad}(g)X \in \overline{\mathbb{F}}X\}$, where $\overline{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$.

Lemma 4.29. *[McN04, Lemma 25] Let S be any maximal torus of $N(X)$. Then there is a unique cocharacter in $X_*(S)$ associated with X.*

Theorem 4.30. *[McN04, Theorem 26] Let $X \in \mathfrak{g}$ be nilpotent. Assume that the G-orbit of X is separable. Then there exists a cocharacter τ associated to X which is defined over $\mathbb{F}$.*

Let τ be a cocharacter associated to X. We define

$$\begin{aligned}
\mathfrak{p}_X &:= \mathfrak{g}(\geq 0;\tau),\\
\mathfrak{n}_X &:= \mathfrak{g}(\geq 1;\tau).
\end{aligned}$$

By [Jan04, Proposition 5.9(a)] the Lie algebra $\mathfrak{p}_X$ is independent of the choice of τ. Hence $\mathfrak{n}_X$ is also independent of the choice of τ.

4.7.2 First proof of Howe's conjecture

All the proofs of Howe's conjecture in this chapter are slight modifications of the proof of Howe's conjecture given in [HC99, Part II]. The proof of Howe's conjecture [HC99, Theorem 12.1] uses the following (sub)sections:

§10.2, §11.1 (excluding Theorem 11.3), §11.3, §12, §13.

These (sub)sections of [HC99], up to and including §12, prove the following: if a lattice L satisfies the condition $C(L)$, then Howe's conjecture holds for L. Here $C(L)$ (see §11.3) is a technical condition depending on L and the nilpotent elements $X \in \mathfrak{g}$ with $|X| = 1$. In §13 it is shown that for every well-adapted lattice L', $C(L')$ holds. Now as stated in [HC99, Remark 10.7]: every lattice L contains a well-adapted lattice L'. Thus Howe's conjecture holds for L' and hence for L.

All those (sub)sections are independent of the results of [HC99, Part I]. In all of them, except §13, the characteristic of $\mathbb{F}$ does not play any role. Only §13 does not generalize verbatim to the case that $\mathbb{F}$ has positive characteristic.

Lemma 4.31. *Assume that G splits over a tamely ramified extension of $\mathbb{F}$.*
Let L be a well-adapted lattice. Let $X_0 \in \mathcal{N} \cap S$. If there exists a cocharacter λ such that

$$X_0 \in \mathfrak{n}(\lambda) \text{ and } \mathfrak{n}(\lambda) \subset [X_0, \mathfrak{g}],$$

then $C(L)$ holds for X_0.

Proof. The proof is basically the same as the proof that $C(L)$ holds for X_0 in [HC99, §13]. We only need to give some modifications to adjust it to the positive characteristic case.

In [HC99, §13.1], the subspaces $\mathfrak{g}_r$ are defined using the semisimple element in an SL_2-triple containing X_0. In our setting, we will use instead $\mathfrak{g}_r = \mathfrak{g}(r; \lambda)$ where λ is a cocharacter associated to X_0. The argument of §13.1 in loc.cit. then proceeds mutatis mutandum using $\lambda' = \lambda^n$ rather than $H_0' = H_0^n$. Now the statements of Lemma 13.2 and Corollary 13.3 of [HC99] hold.

The remaining results of §13.1 are valid in our setting; and – as in [HC99] – they provided a proof of [HC99, Theorem 13.1] – i.e., of Lemma 4.31 of the present text – modulo a proof of [HC99, Lemma 13.5].
The proof of Lemma 13.5 is considered in [HC99, §13.2]; since it depends in part on the exponential mapping, we must adapt this proof.

The only properties of the exponential map used to prove Lemma 13.5 are (3) and (4) in the second paragraph of [HC99, §13.2]. However by going through §13.2 one sees that the following is enough: there exists an open neighborhood U of 0 in $\mathfrak{g}$ and a map $e : U \to G$ such that if $\epsilon > 0$ is small enough:
(3') $\mathfrak{g}(\epsilon) \subset U$, and $K_\epsilon = e(\mathfrak{g}(\epsilon)) \subset K_\gamma$,
(4') there exists a real number $a_3 > 0$ such that for $Z \in \mathfrak{g}(\epsilon)$ and $Y \in \mathfrak{g}$, $|\mathrm{Ad}(e(Z))Y - Y - [Z, Y]| \leq a_3 |Z|^2 |Y|$.
In [HC99], $K_\epsilon(X^\gamma)$ is a subgroup. But that is not required by the proof. Indeed, simply replace the final three lines of the proof on [HC99, p. 69] with the following:

However, $K_\epsilon(X^\gamma)$ is contained in the compact group $K_\gamma(X^\gamma)$. Hence by choosing a subsequence we may assume that $k_n' \to k'$ where $k' \in K_\gamma(X^\gamma)$. Then $p_1(X^{k'\gamma} - X^\gamma) = v$.

Therefore, we can replace the exponential map by a (not-necessarily G-invariant) map between a small open part of $\mathfrak{g}$ and G. Let $e : \mathfrak{g}_{x,0+} \to G_{x,0+}$ be a mock exponential map as constructed in [Adl98, §1.5]. That (3') is satisfied follows immediately from the construction. We will use [Adl98, Proposition 1.6.3] for (4'):
Suppose $r > 0$, $Z \in \mathfrak{g}_{x,r}$ and $Y \in \mathfrak{g}_{x,s}$. Then

$$\mathrm{Ad}(e(Z))Y - Y - [Z, Y] \in \mathfrak{g}_{x,2r+s}.$$

To prove (4') from [Adl98, Proposition 1.6.3] we first need to compare the norm $|\cdot|$ on $\mathfrak{g}$ with the grading $\mathfrak{g}_{x,r}$.
Since $\mathfrak{g}_{x,0}$ is open and compact, there exist constants $c_1, c_2 \in \mathbb{N}$ such that

$$|X| < q^{-c_1} \Rightarrow X \in \mathfrak{g}_{x,0},$$
$$X \in \mathfrak{g}_{x,0} \Rightarrow |X| < c_2.$$

Let ϖ be a uniformizer of $\mathbb{F}$. Then $\varpi^n \mathfrak{g}_{x,0} = \mathfrak{g}_{x,n}$ for all $n \in \mathbb{Z}$.
Assume that $|X| < q^s$. Then $|\varpi^{c_1+\lceil s\rceil} X| < q^{-c_1+s-\lceil s\rceil} < q^{-c_1}$. Thus $\varpi^{c_1+\lceil s\rceil} X \in \mathfrak{g}_{x,0}$. Therefore, $X \in \mathfrak{g}_{x,-c_1-\lceil s\rceil} \subset \mathfrak{g}_{x,-c_1-1-s}$.
Assume that $X \in \mathfrak{g}_{x,s}$. Then $X \in \mathfrak{g}_{x,\lfloor s\rfloor}$. Thus $\varpi^{-\lfloor s\rfloor} X \in \mathfrak{g}_{x,0}$. Therefore, $|\varpi^{-\lfloor s\rfloor} X| < c_2$. So $|X| < c_2 q^{-\lfloor s\rfloor} < c_2 q^{1-s}$.

Let $C_1 := c_1 + 1$ and $C_2 := qc_2$. Then

$$|Z| < r \Rightarrow Z \in \mathfrak{g}_{x,-C_1-\log_q(r)}, \quad \text{and}$$
$$Z \in \mathfrak{g}_{x,r} \Rightarrow |Z| < C_2 q^{-r}.$$

Define $d_1 = |Z|$ and $d_2 = |Y|$. Assume that $|Z|$ is small enough such that $-C_1 - \log_q(d_1) > 0$. Since $Z \in \mathfrak{g}_{x,-C_1-\log_q(d_1)}$ and $Y \in \mathfrak{g}_{x,-C_1-\log_q(d_2)}$, then

$$\mathrm{Ad}(e(Z))Y - Y - [Z, Y] \in \mathfrak{g}_{x,l},$$

where $l = 2(-C_1 - \log_q(d_1)) - C_1 - \log_q(d_2)$. Thus

$$|\mathrm{Ad}(e(Z))Y - Y - [Z, Y]| < C_2\, q^{-l} = C_2\, q^{3C_1} d_1^2 d_2 = C_2\, q^{3C_1} |Z|^2 |Y|.$$

Hence (4') holds for the map $e : \mathfrak{g}_{x,0+} \to G_{x,0+}$ whenever ϵ is small enough. □

Lemma 4.32. *Suppose that* $\mathrm{char}(\mathbb{F})$ *is good for* G *and* $\mathbb{F}$ *is algebraically closed. Let* X *be nilpotent. Let* λ *be a cocharacter associated with* X*. Then*

$$[\mathfrak{g}(-1), X] = \mathfrak{g}(1)$$

and

$$[\mathfrak{n}_X, X] = \mathfrak{g}(\geq 3).$$

Proof. We follow the same line as the proof of [Jan04, Proposition 5.9(c)].
Let G be a group satisfying the standard hypotheses:

1. The derived group of G is simply connected.
2. The characteristic of $\mathbb{F}$ is good for G.

3. There exists a G-invariant nondegenerate bilinear form on $\mathfrak{g}$.

By [Jan04, Proposition 5.8 and Lemma 5.7]

$$[\mathfrak{g}(-1), X] = \mathfrak{g}(1)$$

and

$$[\mathfrak{n}_X, X] = \mathfrak{g}(\geq 3).$$

Now we show that the lemma holds for G if and only if it holds for G_{der}. The cocharacter τ associated to X in G is also the cocharacter τ associated to X in G_{der}. Also $\mathfrak{g}(-1), \mathfrak{g}(1), \mathfrak{g}(\geq 3) \subset \mathfrak{g}'$.

When G is simply connected and the characteristic is very good, then G satisfies the standard hypotheses. The lemma holds for GL_n by [How74, Lemma 2], thus for SL_n as well. Therefore, the lemma holds for all simply connected groups in good characteristic. Hence also for products of those groups.

Let $G = R(G)G_1 \cdots G_m$ with G_i the simple normal connected subgroups of G and $R(G) = Z(G)^\circ$ the radical of G. Let G_i' be the simply connected group belonging to G_i. Let $\pi : R(G)\prod_{i=1}^m G_i' \to G$ be the natural surjective homomorphism. Now $d\pi$ is surjective on the nilpotent elements and maps the associated cocharacter of a nilpotent element to the associated cocharacter of its image. Since the lemma holds for $R(G)\prod_{i=1}^m G_i'$, it also holds for G. □

Theorem 4.33. *Let G be a reductive group which splits over a tamely ramified field extension over $\mathbb{F}$. Assume the characteristic of $\mathbb{F}$ to be good for G. If the nilpotent orbits of G in $\mathfrak{g}$ are separable, then Howe's conjecture holds.*

Proof. As mentioned at the start of this subsection, it is enough to show that $C(L)$ holds for all $X_0 \in \mathcal{N} \cap S$ and well-adapted lattices L.

Let $X_0 \in \mathcal{N} \cap S$.
Let λ be a cocharacter associated with X_0 defined over $\mathbb{F}$ as promised by [McN04, Theorem 26] and the separability of the orbit of X_0. Define $\mathfrak{g}_z := \mathfrak{g}(z; \lambda)$ and

$$\mathfrak{n} = \sum_{r \geq 1} \mathfrak{g}_r, \mathfrak{m} = \mathfrak{g}_0, \overline{\mathfrak{n}} = \sum_{r \geq 1} \mathfrak{g}_{-r}.$$

Now we give a proof of $\mathfrak{n} \subset [X_0, \mathfrak{g}]$.
In the proof of [McN04, Proposition 34]:
"Well, by [Jan04, Proposition 5.9(c)], we have $\overline{\mathrm{Ad}(P)X} = \oplus_{i \geq 2}\mathfrak{g}(i, \phi)$. Since the orbit of X is separable, the differential of the orbit map is surjective."
Thus $\mathfrak{g}(\geq 2) \subset [\mathfrak{g}, X_0]$. Therefore with Lemma 4.32 we have $\mathfrak{n} \subset [X_0, \mathfrak{g}]$.

Thus $C(L)$ holds for all nilpotent elements X_0 with $|X_0| = 1$ by Lemma 4.31. Thus Howe's conjecture holds for G by [HC99, §12]. □

Corollary 4.34. *If G is a simple group which splits over a tamely ramified extension over $\mathbb{F}$ and $\mathbb{F}$ is very good for G, then Howe's conjecture holds.*

Proof. By [RJ67] the nilpotent orbits are separable. □

4.7.3 The case $SO_3(\mathbb{F})$ (char $\mathbb{F} = 2$)

In this subsection char $\mathbb{F} = 2$.

Although there are infinitely many nilpotent conjugacy classes in $SO_3(\mathbb{F})$ and the nilpotent orbits are not separable, Howe's conjecture holds for $SO_3(\mathbb{F})$. We again follow [HC99], but have to make a few more modifications.

The next lemma and its proof are [HC99, Lemma 12.2], with ${}^G\mathfrak{n}$ instead of $\mathcal{N}$.

Lemma 4.35. *Let $\omega \subset \mathfrak{g}$ be a compact set.*
Let S be a split torus and K the stabilizer of 0 in the apartment of S (in the extended building). Take a system of positive roots Φ^+ of (G, S). Let $\mathfrak{n}$ be the Lie algebra for U^+, $\overline{\mathfrak{n}}$ be the Lie algebra for U^- and $\mathfrak{m}$ the Lie algebra of $M := Z_G(S)$.
There is a lattice Λ such that

$$\mathrm{Ad}(G)\omega = \Lambda + \mathrm{Ad}(KS)(\mathfrak{n} \cap \Lambda).$$

Proof. By Bruhat-Tits one has

$$G = KSFK$$

for some finite subgroup F of M.
Since $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{n}$, one has compact subsets $\omega_1, \omega_2, \omega_3$ in $\overline{\mathfrak{n}}$, $\mathfrak{m}$ and $\mathfrak{n}$, respectively, such that

$$\mathrm{Ad}(FK)\omega \subset \omega_1 \oplus \omega_2 \oplus \omega_3.$$

Hence $\mathrm{Ad}(G)\omega \subset \mathrm{Ad}(KS)(\omega_1 \oplus \omega_2 \oplus \omega_3)$.
Now $\mathrm{Ad}(S)\omega_1$ is contained in a compact lattice of $\overline{\mathfrak{n}}$, because $v(\alpha(s)) \geq 0$ for all $\alpha \in \Phi^-$ and $\mathrm{Ad}(S)\omega_2 = \omega_2$. Therefore, there is a lattice L such that

$$\mathrm{Ad}(G)\omega \subset \mathrm{Ad}(K)\left(L + \mathrm{Ad}(S)(\mathfrak{n} \cap L)\right). \qquad \square$$

Since ${}^G\mathfrak{n} = \mathcal{N}$ in characteristic 0, Lemma 12.2 of Harish-Chandra works with $\mathcal{N}$. For the group $SO_3(\mathbb{F})$ this is not the case. Therefore, we shall work with ${}^G\mathfrak{n}$ instead of $\mathcal{N}$.
We start with the definition of $SO_3(\mathbb{F})$.
Define $Q(e_0, e_1, e_2) := e_0^2 + e_1 e_2$.

$$SO_3(\mathbb{F}) := \{g \in GL_3 \mid Q(gv) = Q(v)\}.$$

Let γ be the following cocharacter of SO_3.

$$\gamma(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}.$$

Let T be the following subgroup of SO_3:

$$T := \{\gamma(t) : t \in \mathbb{F}^\times\}.$$

Now T is a maximal torus of SO_3.
The Lie algebra of SO_3 is of the following form:

$$\mathfrak{g} := \left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{F} \right\}.$$

With respect to the cocharacter γ we have a decomposition of the Lie algebra:
$\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ with

$$\mathfrak{n} := \mathfrak{g}(1) = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b \in \mathbb{F} \right\},$$

$$\mathfrak{t} := \mathfrak{g}(0) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} : c \in \mathbb{F} \right\},$$

$$\overline{\mathfrak{n}} := \mathfrak{g}(-1) = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{F} \right\}.$$

Take on $\mathfrak{g}$ the following norm:

$$\left| \begin{pmatrix} 0 & a & b \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \right| = \max(|a|, |b|, |c|).$$

For the extended version of Howe's conjecture, Harish-Chandra needs to consider all nilpotent orbits. But for the regular Howe's conjecture we can restrict ourselves to one nilpotent orbit, namely, the orbit of

$$n := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $N := {}^{G}n \cup \{0\} = {}^{G}\mathfrak{n}$.
For $a, b \in \mathbb{F}$, define the following elements of $\mathfrak{so}_3$ and SO_3:

$$n_{a,b} := \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; u_b := \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & b^2 \\ 0 & 0 & 1 \end{pmatrix}, \; \omega := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Lemma 4.36. *Let $(a, b), (c, d) \in \mathbb{F}^2 - (0, 0)$, then $n_{a,b}$ is in the same conjugacy class as $n_{c,d}$ if and only if there exists $y \in \mathbb{F}$ such that $ab + y^2 = cd$.*
In particular, $N = \{n_{a,b} \mid \exists (y \in \mathbb{F})\; y^2 = ab\}$.

Proof. The conjugation action of the generators of $SO_3(\mathbb{F})$ on the nilpotent elements is as follows:

$$u_c n_{a,b} u_c = n_{a,c^2a+b},$$
$$\gamma(x) n_{a,b} \gamma(x)^{-1} = n_{x^{-1}a,xb},$$
$$\omega n_{a,b} \omega = n_{b,a}.$$

The lemma follows after some calculations. $\square$

Corollary 4.37. *The set N is closed in $\mathfrak{g}$ and $cN = N$ for all $c \in \mathbb{F}^\times$.*

Proof. The nilpotent elements are closed in $\mathfrak{g}$. The function $Q : n_{a,b} \mapsto ab$ is a continuous function from $\mathcal{N}$ to $\mathbb{F}$. Since $\mathbb{F}^{(2)}$ is closed in $\mathbb{F}$, so is $Q^{-1}(\mathbb{F}^{(2)})$. The latter is equal to N by Lemma 4.36. Since closed sets of closed subspaces are closed, N is closed in $\mathfrak{g}$. The second statement is obvious. $\square$

Corollary 4.38. *0 is in the closure of ${}^G n_{a,b}$ if and only if $ab \in \mathbb{F}^{(2)}$.*

Proof. Assume that ab is not a square in $\mathbb{F}$. Let ϖ be a uniformizer of $\mathbb{F}$. Write $ab = \sum_{n=-k}^{\infty} c_n \varpi^n$. Let m be an odd integer such that $c_m \neq 0$, since ab is not a square in $\mathbb{F}$ such m exists. Let $y \in \mathbb{F}^{(2)}$. Write $ab + y = \sum_{n=-l}^{\infty} d_n \varpi^n$, then $d_m = c_n$. Thus $v(ab + y) \leq m$ for every $y \in \mathbb{F}^{(2)}$. Thus every element in the conjugacy class of $n_{a,b}$ is at least at distance q^{-m} from 0. Hence 0 is not in the closure of the G-orbit of $n_{a,b}$.
If ab is a square in $\mathbb{F}$, then either $n_{a,b}$ is conjugate to $n = n_{0,1}$ or $n_{a,b} = 0$. In both cases the closure of their G-orbit contains 0. $\square$

Lemma 4.39. *Let $X \in N$. There is a cocharacter τ such that $X \in \mathfrak{g}(1)$ and $\mathfrak{g}(1) \subset [X, \mathfrak{g}]$.*

Proof. Since these statements are G-invariant, we may and will assume that $X = n_{0,1}$. In this case take $\tau := \gamma$. Clearly $X \in \mathfrak{n} \subset \mathfrak{g}(1)$. Now

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

thus $\mathfrak{n} \subset [X, \mathfrak{g}]$. $\square$

Theorem 4.40. *Howe's conjecture holds in $SO_3(\mathbb{F})$.*

Proof. We follow Harish-Chandra [HC99] again. Recall that the proof of Howe's conjecture is spread out over the following subsections in [HC99]:

§10.2, §11.1 (excluding Theorem 11.3), §11.3, §12, §13.

In order to prove this Theorem we change these subsections by replacing $\mathcal{N}$ by $N = {}^G\mathfrak{n}$. In §11 and §12 three properties of $\mathcal{N}$ are used (between parentheses the Lemmas in [HC99] where the property is used):

1. $\mathcal{N} \cap S$ is compact. (Lemma 11.9 & 12.3)
2. For all compact $\omega \subset \mathfrak{g}$ there exists a lattice L_1, such that ${}^G\omega \subset L_1 + \mathcal{N}$. (Lemma 12.2)
3. If $c \in \mathbb{F}$ and $Y \in \mathcal{N}$, then $cY \in \mathcal{N}$. (Lemma 12.3)

By Corollary 4.37, (1) and (3) also hold for N and (2) is Lemma 4.35. With these modifications §12 shows that we only need to prove $C(L)$ for $X \in N \cap S$ in order to prove Howe's conjecture for L.

Let L be a well-adapted lattice and $X \in N \cap S$. By Lemma 4.39 there exists a cocharacter τ such that

$$X \in \mathfrak{n}(\tau) \text{ and } \mathfrak{n}(\tau) \subset [X, \mathfrak{g}].$$

Thus $C(L)$ holds for X by Lemma 4.31. $\square$

This example shows that the separability of the nilpotent orbits is not a necessary condition for Howe's conjecture to hold.

4.7.4 The case $PGL_n(\mathbb{F})$ with char $\mathbb{F} \mid n$

In this subsection we generalize the results in the previous subsection to the group $PGL_n(\mathbb{F})$. This is the group consisting of the $\mathbb{F}$-points of the algebraic quotient of GL_n by its center of diagonal matrices Z. We have the exceptional isomorphism $PGL_2 \cong SO_3$. Let $G := PGL_n$. We identify $\mathfrak{g}$ with $\mathfrak{gl}_n/\mathfrak{z}$. Now $\mathfrak{gl}_n/\mathfrak{z} := \{X + \mathfrak{z} : X \in \mathfrak{gl}_n\}$. Define $p := \text{char}\, \mathbb{F}$. The nilpotent elements of $\mathfrak{g}$ are exactly those $X + \mathfrak{z}$ such that $X^{p^n} \in \mathfrak{z}$. We define the following G-invariant function ϕ on $\mathcal{N}$: for $X \in \mathfrak{g}(\mathbb{F})$ let $a \in \mathbb{F}$ be such that $X^{p^n} = aI_n$, with I_n the identity matrix. Then $\phi(X + \mathfrak{z}) := a + \mathbb{F}^{(p^n)}$. If $X + \mathfrak{z} = X' + \mathfrak{z}$, then $X - X' \in \mathfrak{z}(\mathbb{F})$. Thus ϕ is well defined.

Lemma 4.41. *The following statements hold for ϕ:*

1. *ϕ is G-invariant.*
2. *$\mathbb{F}^{(p^{n-1})} \subset \text{Im}\, \phi$.*
3. *Let $X + \mathfrak{z}$ be a nilpotent element of $\mathfrak{g}$. Then $\phi(X) \in \mathbb{F}^{(p^n)}$ if and only if there exists a nilpotent matrix $n \in \mathfrak{gl}_n$ such that $n \in X + \mathfrak{z}$.*

Proof. 1. trivial.
2. Let M_x be a block-diagonal matrix consisting of $\frac{n}{p}$ blocks with on each $(p \times p)$-block the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & x \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Then $M_x^p = xI_n$ thus $M_x^{p^n} = x^{p^{n-1}} I_n$.
3. For a nilpotent matrix $n \in \mathfrak{gl}_n$ we have that $\phi(n + \mathfrak{z}) = 0$, thus the only if part is clear. Assume that $X^{p^n} = a^{p^n} I_n$, then $(X - aI_n)^{p_n} = X^{p^n} - a^{p^n} I_n = 0$. Thus $X - aI_n$ is nilpotent. □

Corollary 4.42. *The number of nilpotent orbits is infinite.*

Proof. The group $\mathbb{F}^{(p^{n-1})}/\mathbb{F}^{(p^n)}$ is isomorphic as a group to $\mathbb{F}/\mathbb{F}^{(p)}$, $\mathbb{F}/\mathbb{F}^{(p)}$ is infinite, and $\mathbb{F}^{(p^{n-1})} \subset \text{Im}\, \phi$. □

Thus not all nilpotent orbits are separable. In fact, the orbit of

$$x := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \cdots & & 0 \end{pmatrix},$$

the superdiagonal entries of x are 1, is not separable, since the commutator with

$$x' := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & n & 0 \end{pmatrix},$$

the subdiagonal entries of x' are from left to right equal to $1, 2, \ldots, n$, is equal to (char $\mathbb{F} \mid n$)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(n-1) \end{pmatrix} = \mathrm{Id}_n \in \mathfrak{z}.$$

Lemma 4.43. *If $p \mid n$, then $\{H_\alpha : \alpha \in \Delta\}$ are linearly dependent.*

Proof. Let T be the torus of diagonal matrices. For $i = 1, \ldots, n$, define

$$\epsilon_i \begin{pmatrix} x_1 & 0 & \cdots \\ 0 & \ddots & 0 \\ \cdots & 0 & x_n \end{pmatrix} = x_i.$$

Let $\Delta = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n\}$, then

$$\sum_{i=1}^{n-1} H_{\epsilon_i - \epsilon_{i+1}} = [x, x'] = 0.$$

□

Define $N := \{x + \mathfrak{z} : x \in \mathfrak{gl}_n \mid x \text{ is nilpotent}\}$.

Corollary 4.44. *N is a closed subset of $\mathfrak{g}$ and $cN = N$ for $c \in \mathbb{F}^\times$.*

Proof. The map ϕ is continuous and $0 \in \mathbb{F}/\mathbb{F}^{(p^n)}$ is closed. Thus $N = \phi^{-1}(0)$ is closed in $\mathcal{N}$. Since $\mathcal{N}$ is closed in $\mathfrak{g}$, so is N.
If $x \in \mathfrak{g}$ is nilpotent, then, for all $c \in \mathbb{F}$, cx is also nilpotent. Thus $cN = N$. □

Lemma 4.45. *For every nilpotent element $X \in \mathfrak{gl}_n$ there exists a cocharacter γ, such that $X \in \mathfrak{n}(\gamma)$ and $\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)]$.*

Proof. We follow [How74] and its notation. See page 311 of loc. cit. For $x \in \mathbb{F}$, define the element $\gamma(x) \in M$ to be the transformation which acts on C_i by multiplication by x^i. Then $\mathcal{U} = \mathfrak{n}(\gamma)$. By [How74, Lemma 2] $\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)]$. □

Corollary 4.46. *For every nilpotent element in N, there exists a cocharacter γ such that $X \in \mathfrak{n}(\gamma)$ and $\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)]$.*

Proof. Let $X \in GL_n$ and let $\gamma \in X_*(G)$ be the cocharacter of Lemma 4.45. Let $\varphi : GL_n \to PGL_n$ be the natural homomorphism. Since $\mathrm{d}\varphi$ is surjective and $\mathrm{d}\varphi(\mathrm{Ad}(x)X) = \mathrm{Ad}(\varphi(x))\mathrm{d}\varphi(X)$, we have $\mathfrak{n}(\varphi\gamma) = \mathrm{d}\varphi\left(\mathfrak{n}(\gamma)\right)$ and $\mathfrak{p}(\varphi\gamma) = \mathrm{d}\varphi\left(\mathfrak{p}(\gamma)\right)$. We conclude that $\varphi\gamma$ is the desired cocharacter for $X + \mathfrak{z}$. □

Theorem 4.47. *Howe's conjecture holds in PGL_n.*

Proof. We follow Harish-Chandra [HC99] again and mention the adjustments. Just like in the SO_3 case we replace $\mathcal{N}$ by $N = {}^G\mathfrak{n}$. The proof of Harish-Chandra uses three properties of $\mathcal{N}$ (between parentheses the Lemmas in [HC99] where the property is used):

1. $\mathcal{N} \cap S$ is compact. (Lemma 11.9 & 12.3)
2. For all compact $\omega \subset \mathfrak{g}$ there exists a lattice L_1, such that ${}^G\omega \subset L_1 + \mathcal{N}$. (Lemma 12.2).
3. If $c \in \mathbb{F}$ and $Y \in \mathcal{N}$, then $cY \in \mathcal{N}$. (Lemma 12.3)

By Corollary 4.44, (1) and (3) also hold for N and (2) is Lemma 4.35.

Let L be a well-adapted lattice and $X \in N \cap S$. By Corollary 4.46 and Lemma 4.31 $C(L)$ holds for X. Thus Howe's conjecture holds in PGL_n. □

4.7.5 The Howe's conjecture classification ($\mathbb{F}$-split case)

In this subsection we determine exactly for which $\mathbb{F}$-split reductive groups Howe's conjecture holds.

Lemma 4.48. *Let G be a $\mathbb{F}$-split group. If T^{ad} is an $\mathbb{F}$-split torus of G^{ad}, then $\mathrm{Ad}^{-1}(T^{\mathrm{ad}})$ is an $\mathbb{F}$-split torus of G.*

Proof. Without loss of generality we assume that T^{ad} is a maximal $\mathbb{F}$-split torus of G^{ad}. Let S be a maximal split torus of G and B a Borel subgroup containing S. Then $S^{\mathrm{ad}} := \mathrm{Ad}(S)$ is a maximal split torus of G^{ad} and B^{ad} a Borel subgroup containing S^{ad}. Take $g \in G^{\mathrm{ad}}(\mathbb{F})$ such that $gS^{\mathrm{ad}}g^{-1} = T^{\mathrm{ad}}$. Take $w^{\mathrm{ad}} \in W^{\mathrm{ad}}$ such that $g \in U_{(w^{\mathrm{ad}})^{-1}}w^{\mathrm{ad}}B^{\mathrm{ad}}$. By multiplying g with a suitable element of S^{ad}, we may assume that $g \in U_{(w^{\mathrm{ad}})^{-1}}w^{\mathrm{ad}}U^{\mathrm{ad}}$. Take $w \in W$ such that $\mathrm{Ad}(w) = w^{\mathrm{ad}}$, then

$$\mathrm{Ad} : U_{w^{-1}}wU \to U_{(w^{\mathrm{ad}})^{-1}}w^{\mathrm{ad}}U^{\mathrm{ad}}$$

is a bijection. Therefore, there exists $h \in G(\mathbb{F})$ such that $\mathrm{Ad}(h) = g$. Thus

$$\mathrm{Ad}(hSh^{-1}) = gS^{\mathrm{ad}}g^{-1} = T^{\mathrm{ad}}.$$

Thus $\mathrm{Ad}^{-1}(T^{\mathrm{ad}}) = hSh^{-1}$ is an $\mathbb{F}$-split torus. □

Theorem 4.49. *Let G be a reductive $\mathbb{F}$-split group, then the following statements are equivalent*

1. *The characteristic p of $\mathbb{F}$ is good and $p \nmid \kappa_v(G)$.*

2. *For all compact subsets ω and lattices L in $\mathfrak{g}$:*

$$\dim J_L(\omega) < \infty.$$

Proof. If the characteristic p of $\mathbb{F}$ is bad, then G has bad pairs. So in that case Howe's conjecture does not hold.
If $p \mid \kappa_v(G)$, then Howe's conjecture does not hold by Theorem 4.28.

Assume that p is good and $p \nmid \kappa_v(G)$. We will use the proof of Howe's conjecture given in [HC99, Part II].
Let A be a maximal $\mathbb{F}$-split torus of G and B a Borel subgroup containing A. Let $\mathfrak{n}$ be the Lie algebra of the unipotent radical of B. Let $N := {}^G\mathfrak{n}$. For the moment assume that N has the following properties (between parentheses the corresponding Lemmas and section in [HC99]):

1. For all compact $\omega \subset \mathfrak{g}$ there exists a lattice L_1 such that ${}^G\omega \subset L_1 + N$. (Lemma 12.2)
2. For all $X \in N$ there exists an $\mathbb{F}$-rational cocharacter $\gamma \in X_*(G)$ such that $X \in \mathfrak{n}(\gamma)$ and $\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)]$. (§13.1)
3. N is closed. (Lemma 11.9 & 12.3)
4. If $c \in \mathbb{F}$ and $Y \in N$, then $cY \in N$. (Lemma 12.3)

By (1),(3),(4) and [HC99, §11.1 & §12], if $C(L)$ holds for all $X_0 \in N \cap S$, then Howe's conjecture holds for L. For $X_0 \in N \cap S$ and well-adapted lattices L, $C(L)$ holds by (2) and Lemma 4.31. Since every lattice contains a well-adapted lattice, Howe's conjecture holds for all lattices L. Thus it is enough to show that N has these four properties.

Property (1) is Lemma 4.35.

Let $G = R(G)G_1 \cdots G_m$ with G_i connected normal simple groups and $R(G)$ the radical of G. Let $G^{\mathrm{ad}} = \prod_{i=1}^m G_i^{\mathrm{ad}}$ be the adjoint group of G. Since p is good for G, it is also good for all the groups G_i^{ad}. Either p is very good for G_i^{ad} or $G_i^{\mathrm{ad}} = PGL_n$ with $p \mid n$. Thus by Corollary 4.46 and the proof of Corollary 4.34 for all $X_i \in {}^{G_i^{\mathrm{ad}}}\mathfrak{n}_i^{\mathrm{ad}}$ there exists a cocharacter γ_i^{ad} defined over $\mathbb{F}$ such that $X_i \in \mathfrak{n}_i^{\mathrm{ad}}(\gamma_i^{\mathrm{ad}})$ and $\mathfrak{n}_i^{\mathrm{ad}}(\gamma_i^{\mathrm{ad}}) = [X_i, \mathfrak{p}_i^{\mathrm{ad}}(\gamma_i^{\mathrm{ad}})]$. Since G^{ad} is a direct product of G_i^{ad}, for every $X \in {}^{G^{\mathrm{ad}}}\mathfrak{n}^{\mathrm{ad}}$ there exists a character γ such that $X \in \mathfrak{n}(\gamma)$ and $\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)]$.

Property (2) will be shown for $X \in \mathcal{N} \cap \mathrm{d}(\mathrm{Ad})^{-1}({}^{G^{\mathrm{ad}}}\mathfrak{n}^{\mathrm{ad}}) \supset N$.
Let $X \in \mathcal{N} \cap \mathrm{d}(\mathrm{Ad})^{-1}({}^{G^{\mathrm{ad}}}\mathfrak{n}^{\mathrm{ad}})$. Let $X^{\mathrm{ad}} := \mathrm{d}(\mathrm{Ad})(X)$. Let γ^{ad} be a cocharacter of G^{ad} such that $X^{\mathrm{ad}} \in \mathfrak{n}^{\mathrm{ad}}(\gamma^{\mathrm{ad}})$ and $\mathfrak{n}^{\mathrm{ad}}(\gamma^{\mathrm{ad}}) = [X^{\mathrm{ad}}, \mathfrak{p}^{\mathrm{ad}}(\gamma^{\mathrm{ad}})]$. Let T be a maximal split torus of G such that γ^{ad} is a cocharacter of $T^{\mathrm{ad}} = \mathrm{Ad}(T)$. Take γ a cocharacter of T and $m \in \mathbb{N}_{>0}$ such that $\mathrm{Ad} \circ \gamma = m\gamma^{\mathrm{ad}}$. Let $k \in \mathbb{N}_{>0}$ be such that $\mathrm{Ad}(\gamma^{\mathrm{ad}}(t))X^{\mathrm{ad}} = t^k X^{\mathrm{ad}}$ for all $t \in \mathbb{F}^\times$. Then

$$\mathrm{Ad}(\gamma(t))X = t^{km}X + X_z,$$

for some $X_z \in \mathfrak{z}$. Since $\mathrm{Ad}(\gamma(t))X$ and $t^{km}X$ are nilpotent, $X_z = 0$. Thus $X \in \mathfrak{n}(\gamma)$. Since $p \nmid k_v(G)$, $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g} \to \mathfrak{g}^{\mathrm{ad}}$ is surjective by Proposition 4.10. Hence $\mathrm{d}(\mathrm{Ad}) : \mathfrak{p}(\gamma) \to \mathfrak{p}^{\mathrm{ad}}(\gamma^{\mathrm{ad}})$ is surjective. Also $\mathrm{d}(\mathrm{Ad}) : \mathfrak{n}(\gamma) \to \mathfrak{n}^{\mathrm{ad}}(\gamma^{\mathrm{ad}})$ is a bijection. Thus

$$\mathfrak{n}(\gamma) = [X, \mathfrak{p}(\gamma)],$$

because $\mathfrak{n}^{\mathrm{ad}}(\gamma^{\mathrm{ad}}) = [X^{\mathrm{ad}}, \mathfrak{p}^{\mathrm{ad}}(\gamma^{\mathrm{ad}})]$. Therefore, for all $X \in \mathcal{N} \cap \mathrm{d}(\mathrm{Ad})^{-1}({}^{G^{\mathrm{ad}}}\mathfrak{n}^{\mathrm{ad}})$ property (2) holds.

Moreover, since $X \in \mathfrak{n}(\gamma)$, also $X \in N$. Therefore,

$$N = \mathcal{N} \cap \mathrm{d}(\mathrm{Ad})^{-1}({}^{G^{\mathrm{ad}}}\mathfrak{n}^{\mathrm{ad}}).$$

Since $\mathcal{N}$ and ${}^{G^{\mathrm{ad}}}\mathfrak{n}^{\mathrm{ad}}$ are closed and $\mathrm{d}(\mathrm{Ad})$ is continuous, also N is closed (property (3)).

Because $c\mathfrak{n} \subset \mathfrak{n}$ for all $c \in \mathbb{F}$, property (4) follows. □

Corollary 4.50. *If G is $\mathbb{F}$-split and has finitely many nilpotent orbits, then Howe's conjecture holds for G.*

Proof. If the characteristic p of $\mathbb{F}$ is bad for G or if $p \mid \kappa_v(G)$, then there are infinitely many nilpotent orbits. □

4.8 The separable classification

In this section $\mathbb{F}$ is an algebraically closed field of characteristic p.

In this section we give a characterization of the reductive groups whose nilpotent orbits are all separable. As a consequence we get a large class of reductive groups for which the number of nilpotent orbits is finite and Howe's conjecture holds. We take a look at the cokernels of the following functions:

$$\Phi : X_*(T) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z}),$$
$$\Phi^\vee : X^*(T) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta^\vee, \mathbb{Z}).$$

Lemma 4.51. *$p \mid \rho_v(G)$ if and only if the $H_\alpha := \mathrm{d}\alpha^\vee(1) \in \mathfrak{t}$, for $\alpha \in \Delta$, are linearly dependent.*

Proof. We have the following isomorphism of vector spaces: $\mathfrak{t} \cong X_*(T) \otimes_{\mathbb{Z}} \mathbb{F}$. Let ϵ_i be a basis for $X^*(T)$ and $\epsilon_i^\vee$ a dual basis in $X_*(T)$. Let $\alpha^\vee \in X_*(T)$. Now $\alpha^\vee = \sum_{i=1}^m \langle \epsilon_i, \alpha^\vee \rangle \epsilon_i^\vee$. Hence $\mathrm{d}\alpha^\vee(1) = \sum_{i=1}^m \langle \epsilon_i, \alpha^\vee \rangle \mathrm{d}\epsilon_i^\vee(1)$. Let $\alpha_1^\vee, \ldots, \alpha_n^\vee$ be the simple roots in $\Delta^\vee$. Define M to be the $n \times m$ matrix with the following entries

$$M_{ij} := \langle \epsilon_j, \alpha_i^\vee \rangle .$$

Then M is the matrix corresponding to the map $\Phi^\vee$.
The matrix M^{tr} is the matrix corresponding to the linear span of the H_{α_i}'s.
Let $(d_1, \ldots, d_n)$ be the entries on the diagonal of the Smith normal form of M. Then $\rho_v(G) = \#\mathrm{coker}\, \Phi^\vee = \prod_{i=1}^n d_i$. The linear span of the H_{α_i}'s is n-dimensional if and only if $p \nmid \prod_{i=1}^n d_i$. □

Theorem 4.52. *The nilpotent orbits are separable if and only if the p is good and $p \nmid \kappa_v(G)$ and $p \nmid \rho_v(G)$.*

When G is semisimple this is [Spr66, Theorem 5.9].

Proof ⇒. If p is bad or divides $\kappa_v(G)$, then the regular nilpotent orbit is inseparable by Corollary 4.21 and Theorem 4.8. Assume that p divides $\rho_v(G)$. Let $X := \sum_{\alpha\in\Delta} E_\alpha$. Then:

$$\left[\sum_{\alpha\in\Delta} E_\alpha, \sum_{\alpha\in-\Delta} c_\alpha E_\alpha\right] = \sum_{\alpha\in\Delta} c_\alpha H_\alpha.$$

Now p divides the cokernel exactly when the $H_\alpha = \mathrm{d}\alpha^\vee(1)$ are linearly dependent. Thus there exists $Y \in \mathfrak{n}_{-1} - \{0\}$, such that $[X, Y] = 0$. Since $Z_G(X) \subset B$ and $\mathfrak{n}_{-1} \cap \mathfrak{b} = 0$, the orbit of X is not separable. □

Before we prove the implication in the other direction, we first state a few lemmas.

Lemma 4.53. *If p is good for G, then*

$$\mathfrak{g} = \mathfrak{g}_A \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$$

with $G = G_A \prod_{i=1}^m G_i$ where G_i are all the closed normal connected simple groups not of type A and G_A is generated by $R(G)$ and the closed normal connected simple groups of type A in G.

Proof. We have $\mathrm{Ad}(G) \cong G_A^{\mathrm{ad}} \times G_1^{\mathrm{ad}} \times \cdots \times G_m^{\mathrm{ad}}$. Let $\Pi : \mathrm{Ad}(G) \to G_1^{\mathrm{ad}} \times \cdots \times G_m^{\mathrm{ad}}$ be the corresponding projection map. Since p is good for G, it is very good for $G_c = G_1 \cdots G_m$. Thus the linear map $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g}_c \to \mathfrak{g}_c^{\mathrm{ad}}$ is surjective. Since $\dim G_c = \dim \mathrm{Ad}(G_c)$, it is a bijection. Thus $\mathrm{d}(\Pi \circ \mathrm{Ad}) : \mathfrak{g}_c \to \mathfrak{g}_c^{\mathrm{ad}}$ is a bijection. Therefore, $\mathfrak{g}_c = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$, since p is very good for G_c. Moreover, $\ker \mathrm{d}(\Pi \circ \mathrm{Ad}) \cap \mathfrak{g}_c = 0$. Since $\mathfrak{g}_A \subset \ker \mathrm{d}(\Pi \circ \mathrm{Ad})$, also $\mathfrak{g}_A \cap \mathfrak{g}_c = 0$. Because $\dim \mathfrak{g}_A + \dim \mathfrak{g}_c = \dim \mathfrak{g}$, the Lemma follows. □

Lemma 4.54. *If p is good for G and $p \nmid \kappa_v(G)$, then $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g}_A \to \mathfrak{g}_A^{\mathrm{ad}}$ is surjective.*

Proof. By Lemma 4.53 and its proof we have $\mathfrak{g} = \mathfrak{g}_A \oplus \mathfrak{g}_c$, and $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g}_c \to \mathfrak{g}_c^{\mathrm{ad}}$ is surjective. Since $p \nmid \kappa_v(G)$ the map $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g} \to \mathfrak{g}_A^{\mathrm{ad}} \oplus \mathfrak{g}_c^{\mathrm{ad}}$ is surjective. Let $\Pi_A : \mathrm{Ad}(G) \to G_A^{\mathrm{ad}}$, then $\mathrm{d}(\Pi_A \circ \mathrm{Ad}) : \mathfrak{g} \to \mathfrak{g}_A^{\mathrm{ad}}$ is surjective. Since $\mathfrak{g}_c$ is contained in its kernel and $\mathfrak{g} = \mathfrak{g}_A \oplus \mathfrak{g}_c$, $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g}_A \to \mathfrak{g}_A^{\mathrm{ad}}$ is surjective. □

Corollary 4.55. *If p is good for G and $p \nmid \kappa_v(G)$, then p does not divide the order of the cokernel of the following map:*

$$\Phi_A : X_*(T_A) \to \mathrm{Hom}_Z(\mathbb{Z}\Delta_A, \mathbb{Z}).$$

Proof. The map $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g}_A \to \mathfrak{g}_A^{\mathrm{ad}}$ is surjective, thus $p \nmid \#\mathrm{coker}\, \Phi_A$ by Proposition 4.10. □

Lemma 4.56. *Let $n = n_1 + \cdots + n_m$ and γ_i be the cocharacters $\gamma_i \in X_*(T \cap G_i^{\mathrm{ad}})$ associated with n_i in $\mathfrak{g}_i^{\mathrm{ad}}$. Let $\gamma \in X_*(T)$ be the cocharacter associated with n in G. Then* $\mathrm{d}(\mathrm{Ad}) \circ \gamma = \sum_{i=1}^m \gamma_i$.

Proof. Clearly, $\sum_{i=1}^m \gamma_i$ is a cocharacter associated with n in $\mathfrak{g}^{\mathrm{ad}}$. Moreover $\mathrm{d}(\mathrm{Ad}) \circ \gamma$ is a cocharacter associated with n in $\mathfrak{g}^{\mathrm{ad}}$ (see [Jan04, §5.6]). Since there is at most one cocharacter of $\mathrm{Ad}(T)$ associated with n by [McN04, Corollary 22], they are equal. □

Lemma 4.57. *Let $G = GL_m$ and $G^{\mathrm{ad}} = PGL_m$. Let $n \in \mathfrak{g}^{\mathrm{ad}}$ be a nilpotent element with associated cocharacter γ. Then*

$$\begin{aligned} &[n, \mathfrak{g}^{ad}(k)] = \mathfrak{g}^{ad}(k+2), && \textit{for } k \geq -1, \\ &[n, \cdot] : \mathfrak{g}^{ad}(k) \to \mathfrak{g}^{ad}(k+2) \textit{ is injective}, && \textit{for } k = -1 \textit{ and } k \leq -3. \end{aligned}$$

Proof. For GL_m and $n \in \mathfrak{g}_m$ nilpotent, $[n, \mathfrak{g}(k)] = \mathfrak{g}(k+2)$ for $k \geq -1$ and $[n, \cdot] : \mathfrak{g}(k) \to \mathfrak{g}(k+2)$ is injective for $k \leq -1$. Since the map

$$\mathrm{d(Ad)} : \bigoplus_{k \geq 1} \mathfrak{g}(-k) \oplus \mathfrak{g}(k) \to \bigoplus_{k \geq 1} \mathfrak{g}^{\mathrm{ad}}(-k) \oplus \mathfrak{g}^{\mathrm{ad}}(k)$$

is a bijection, $[n, \mathfrak{g}^{\mathrm{ad}}(k)] = \mathfrak{g}^{\mathrm{ad}}(k+2)$ for $k \geq -1$ and $[n, \cdot] : \mathfrak{g}^{\mathrm{ad}}(k) \to \mathfrak{g}^{\mathrm{ad}}(k+2)$ is injective for $k = 1$ and $k \geq -3$. □

Lemma 4.58. *Let $G = GL_m$. Let $n = \sum_{\alpha \in \Gamma} c_\alpha n_\alpha$, with $\Gamma \subset \Delta$. If $[n, m] \in \mathfrak{z}$ and $m \in \mathfrak{g}(-2)$, then $m = \sum_{\alpha \in -\Gamma} d_\alpha n_\alpha$ for some $d_\alpha \in \mathbb{F}$.*

Proof. Let $\gamma \in X_*(T)$ be such that there exists a $l \in \mathbb{N}_{\geq 0}$ such that for all $\alpha \in \Delta$:

$$\langle \alpha, \gamma \rangle = \begin{cases} l & \text{if } \alpha \in \Gamma, \\ 0 & \text{if } \alpha \notin \Gamma. \end{cases}$$

We know that $[n, \cdot] : \mathfrak{g}(-2; \tau) \to \mathfrak{g}(0; \tau)$ is injective for the associated cocharacter $\tau \in X_*(T)$ of n. Define $\mathfrak{g}(-2) := \mathfrak{g}(-2; \tau)$ and $\mathfrak{g}_i(-2) := \mathfrak{g}(-2; \tau) \cap \mathfrak{g}(il; \gamma)$. Then

$$\mathfrak{g}(-2) = \bigoplus \mathfrak{g}_i(-2)$$

and

$$[n, \mathfrak{g}_i(-2)] \subset \mathfrak{g}(0; \tau) \cap \mathfrak{g}(l(i+1); \gamma).$$

Because $\mathfrak{z} \subset \mathfrak{g}(0; \gamma)$ and $[n, \cdot]|_{\mathfrak{g}(-2)}$ is injective, then $m \in \mathfrak{g}(-l; \gamma)$. □

Proof of Theorem 4.52 $\Leftarrow$. Let $n \in \mathfrak{n}$. Take $n_A \in \mathfrak{n}_A$ and $n_i \in \mathfrak{n}_i$, such that

$$n = n_A + n_1 + \cdots + n_m.$$

Then

$$Z_G(n) = Z_{G_A}(n_A) \prod_{i=1}^{m} Z_{G_i}(n_i),$$

$$Z_{\mathfrak{g}}(n) = Z_{\mathfrak{g}_A}(n_A) \oplus \bigoplus_{i=1}^{m} Z_{\mathfrak{g}_i}(n_i).$$

Since the G_i are simple and p is very good for G_i, the G_i-orbit of n_i is separable:

$$\dim Z_{\mathfrak{g}_i}(n_i) = \dim Z_{G_i}(n_i).$$

Thus we are left with showing that $\dim Z_{G_A}(n_A) = \dim Z_{\mathfrak{g}_A}(n_A)$. Since p is good for G, it is also good for G_A. By Corollary 4.55 and Lemma 4.51, p does not divide the order of the cokernels of Φ_A and $\Phi_A^\vee$.

Thus, without loss of generality, we assume that G only consists of groups of type A and a center. Thus $G^{\mathrm{ad}} = \prod_{i=1}^{k} PGL_{n_i}$.

Since $p \nmid \kappa_v(G)$, the map $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g} \to \mathfrak{g}^{\mathrm{ad}}$ is surjective. Let $n \in \mathfrak{g}$ be nilpotent and let γ be a cocharacter associated with n. Define $P := P(\gamma)$. Then $\mathrm{Ad} \circ \gamma$ is a cocharacter associated with $\mathrm{Ad}(n)$. For G^{ad} the following holds:

$$\begin{aligned} &[n, \mathfrak{g}^{\mathrm{ad}}(k)] = \mathfrak{g}^{\mathrm{ad}}(k+2), && \text{for } k \geq -1, \\ &[n, \cdot] : \mathfrak{g}^{\mathrm{ad}}(k) \to \mathfrak{g}^{\mathrm{ad}}(k+2) \text{ is injective}, && \text{for } k = -1 \text{ and } k \leq -3. \end{aligned}$$

Since $\mathrm{d}(\mathrm{Ad})$ is surjective and injective on the nilpotent elements, then

$$\begin{aligned} &[n, \mathfrak{p}] = \mathfrak{g}(\geq 2), \\ &[n, \cdot] : \mathfrak{g}(k) \to \mathfrak{g}(k+2) \text{ is injective for } k = -1 \text{ and } k \leq -3. \end{aligned}$$

Now,

$$\dim Z_G(n) = \dim Z_P(n) = \dim Z_{\mathfrak{p}}(n),$$

because $\overline{\mathrm{Ad}\, P(n)} = \mathfrak{g}(\geq 2)$ and $[n, \mathfrak{p}] = \mathfrak{g}(\geq 2)$.
If $Z_{\mathfrak{g}}(n) \cap \mathfrak{g}(k) = 0$ for $k \leq -1$, then $Z_{\mathfrak{g}}(n) = Z_{\mathfrak{p}}(n)$.
For $k = -1$ and $k \leq -3$ the function $[n, \cdot] : \mathfrak{g}(k) \to \mathfrak{g}(k+2)$ is injective. Thus $Z_{\mathfrak{g}}(n) \cap \mathfrak{g}(k) = 0$, for $k = -1$ and $k \leq -3$. Thus we only need to prove that the kernel of $[n, \cdot] : \mathfrak{g}(-2) \to \mathfrak{g}(0)$ is 0.

In G^{ad} every nilpotent element is conjugate to an element of the form $\sum_{\alpha \in \Gamma} E_\alpha$, with $\Gamma \subset \Delta$. Let $n = \sum_{\alpha \in \Gamma} E_\alpha$ with $\Gamma \subset \Delta$ and $m \in \mathfrak{g}(-2)$. If $[n, m] = 0$, then $[\mathrm{d}(\mathrm{Ad})(n), \mathrm{d}(\mathrm{Ad})(m)] = 0$. By Lemma 4.58, then $m = \sum_{\alpha \in -\Gamma} c_\alpha E_\alpha$ for some $c_\alpha \in \mathbb{F}$. Now

$$0 = [n, m] = \sum_{\alpha \in \Delta} c_\alpha H_\alpha.$$

Since $p \nmid \rho_v(G)$, the H_α are linearly independent. Thus $c_\alpha = 0$ for all $\alpha \in -\Gamma$, hence $m = 0$. Thus the kernel of $[n, \cdot] : \mathfrak{g}(-2) \to \mathfrak{g}(0)$ is 0. □

4.9 On the number of nilpotent orbits

In this section, we discuss when the number of nilpotent orbits is finite.

Theorem 4.59. *[McN04, Theorem 40] If p is good and all the nilpotent orbits are separable, then there are only finitely many nilpotent orbits.*

Corollary 4.60. *If p is good and $p \nmid \kappa_v(G)$ and $p \nmid \rho_v(G)$, then there are only finitely many nilpotent orbits.*

Proof. The condition in the corollary is equivalent to the one in Theorem 4.59 by Theorem 4.52. □

In this section we will prove the converse of Corollary 4.60. If G is $\mathbb{F}$-split and p is bad or divides $\kappa_v(G)$, then there are infinitely many regular nilpotent orbits by Theorem 4.26 and Proposition 4.6. So it is enough to prove that if G is $\mathbb{F}$-split, p is good, $p \nmid \kappa_v(G)$ and $p | \rho_v(G)$, then G has infinitely many nilpotent orbits. First a theorem that we can easily deduce from the theory of the previous section.

Theorem 4.61. *If G is semisimple and the characteristic of $\mathbb{F}$ is not very good, then there are infinitely many nilpotent orbits.*

Proof. If the characteristic of $\mathbb{F}$ is bad, then we have already showed that there are infinitely many nilpotent orbits. So without loss of generality we assume G has at least one normal simple group of type A_n, with $p|n+1$. Now the proof is split into two cases: $p\,|\kappa_v(G)$ and $p\,\not|\kappa_v(G)$.

If $p\,|\kappa_v(G)$, then $\mathfrak{g}$ has infinitely many nilpotent orbits by Proposition 4.6.

If $p\,\not|\kappa_v(G)$, then $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g} \to \mathfrak{g}^{\mathrm{ad}}$ is an isomorphism by Theorem 4.10. Since there are infinitely many nilpotent orbits in $\mathfrak{g}^{\mathrm{ad}}$ by Corollary 4.42, there are also infinitely many nilpotent orbits in $\mathfrak{g}$. $\square$

Proposition 4.62. *Let G be a reductive group with only normal simple subgroups of type A for which p is not very good. Assume that $p \mid \rho_v(G)$. Let H be a reductive group with $G \triangleleft H$. Let $\mathcal{N}$ be the set of nilpotent elements of $\mathfrak{g}$. Then there are infinitely many nilpotent H-orbits in $\mathcal{N}$.*

Proof. The proof of this proposition is distributed over two lemmas.

Lemma 4.63. *If $\alpha^\vee \in \Delta^\vee$, then $\mathrm{Ad} \circ \alpha^\vee \in \Delta^\vee_{\mathrm{ad}}$.*

Proof. The reader could verify this by taking the Chevalley basis on $\mathfrak{g}$. $\square$

Let $\Delta^\vee = \{\alpha^\vee_{11}, \ldots \alpha^\vee_{nm_n}\}$, such that $\alpha^\vee_{ij}$ is connected in the Dynkin diagram with $\alpha^\vee_{i'j'}$ if and only if $i = i'$ and $j = j+1$.

Lemma 4.64. *If $\sum_{\alpha\in\Delta} c_\alpha \mathrm{d}\alpha^\vee(1) = 0$, then for every i there exists a c_i such that $c_{\alpha_{ij}} = jc_i$ for all j.*

Proof. Because $\sum_{\alpha\in\Delta} c_\alpha \mathrm{d}\alpha^\vee(1) = 0$, also

$$\sum_{\alpha\in\Delta_{\mathrm{ad}}} c_\alpha \mathrm{d}(\mathrm{Ad} \circ \alpha^\vee)(1) = 0$$

($\mathrm{d}(\mathrm{Ad})\,(\mathrm{d}\alpha^\vee(1)) = \mathrm{d}(\mathrm{Ad} \circ \alpha^\vee)(1)$).
Since $\mathfrak{g}^{\mathrm{ad}} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, with $\mathfrak{g}_i$ the Lie algebra of PGL_{m_i+1}, then for every i:

$$\sum_{j=1}^{m_i} c_{\alpha_{ij}} \mathrm{d}(\mathrm{Ad} \circ \alpha^\vee_{ij})(1) = 0.$$

A small calculation in $\mathfrak{g}_i$ shows that there exists a c_i such that $c_{\alpha_{ij}} = jc_i$. $\square$

Since $p|\rho_v(G)$, by Lemma 4.64 there exist $c_i \in \mathbb{F}$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} jc_i \mathrm{d}\alpha^\vee_{ij}(1) = 0$$

and at least one of the $c_i \neq 0$. Without loss of generality assume that $1, \ldots, k$ are the i with $c_i \neq 0$.

Let $i \leq k$. Let $M_i(x)$ be the block matrix consisting of $\frac{m_i}{p}$ blocks of $p \times p$-matrices, with on each block the following matrix

$$\begin{pmatrix} 0 & \cdots & 0 & c_i x \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \ddots & 0 & 1 & 0 \end{pmatrix}.$$

Thus the entries of M_i are as follows:

$$(M_i)_{kl} := \begin{cases} 1 & \text{if } k = l+1 \text{ and } p \not| \, l, \\ c_i x & \text{if } l = k+p-1 \text{ and } p \mid l, \\ 0 & \text{otherwise.} \end{cases}$$

Then $M_i(x)^p = c_i x I_{n_i}$.
Let $N(x)$ be the element in $\mathfrak{g}$ corresponding with $M_1(x) \oplus \cdots \oplus M_k(x)$. Then

$$N(x)^p = \sum_{i=1}^{n} \sum_{j=1}^{m_i} x j c_i \mathrm{d}\alpha_{ij}^{\vee}(1) = 0.$$

Thus $N(x)$ is nilpotent.
Let q be a power of p such that $\mathcal{N}^{\mathrm{ad}} = \{X \in \mathfrak{g}^{\mathrm{ad}} \mid X^q = 0\}$.
Let $\phi' : \mathcal{N}^{\mathrm{ad}} \to \mathbb{F}/\mathbb{F}^q$ be the following function:
Take $X_i \in \mathfrak{gl}_{m_i+1}$ such that $X = \oplus_{i=1}^{n} X_i + \mathfrak{z}_i$. Then for each i we have a z_i such that $X_i^q = z_i I_{m_i+1}$ in $\mathfrak{gl}_{m_i+1}$. Define $\phi'(X) := z_1$.
If X_i' are also representatives for X, then $z_i' = z_i + a_i^q$ for $a_i \in \mathbb{F}$. Thus ϕ' is well-defined. Since $G^{\mathrm{ad}} = \prod_{i=1}^{n} PGL_{m_i+1}$, ϕ' is also G^{ad}-invariant. Define $\phi : \mathcal{N} \to \mathbb{F}/\mathbb{F}^q$ by $\phi := \phi' \circ \mathrm{d}(\mathrm{Ad})$. The function ϕ is H-invariant, because H acts on $\mathcal{N}$ by conjugation and $G_1 \lhd H$. For $x \in \mathbb{F}$,

$$\phi(N(x)) = x^{\frac{q}{p}}.$$

Since $\mathbb{F}^{\frac{q}{p}}/\mathbb{F}^q \cong \mathbb{F}/\mathbb{F}^{(p)}$ is infinite and ϕ is H-invariant, there are infinitely many nilpotent H-orbits. $\square$

Theorem 4.65. *If p is good and $p \not| \kappa_v(G)$, but $p | \rho_v(G)$, then there are infinitely many nilpotent orbits.*

Proof. Let $G = R(G) G_1 \cdots G_l$ with G_i the minimal simple normal connected subgroups of G and $R(G)$ the radical of G. Assume that $G_1, \ldots, G_n$ are the groups of type A for which p is not very good. Define $G_A := R(G) G_1 \cdots G_n$ and $G_C := G_{n+1} \cdots G_l$. Because $p \not| \kappa_v(G)$, the map $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g} \to \mathfrak{g}^{\mathrm{ad}}$ is surjective. Because p is very good for G_C, $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g}_C \to \mathfrak{g}_C^{\mathrm{ad}}$ is surjective. Since G_C is semisimple, the map is even an isomorphism.

$$\mathfrak{g}^{\mathrm{ad}} \cong \bigoplus_{i=1}^{l} \mathfrak{g}_i^{\mathrm{ad}} = \mathfrak{g}_A^{\mathrm{ad}} \oplus \mathfrak{g}_C^{\mathrm{ad}}.$$

Define $\mathrm{Ad}_C : \mathfrak{g} \to \mathfrak{g}_C^{\mathrm{ad}}$ by the composition of the projection and $\mathrm{d}(\mathrm{Ad}) : \mathfrak{g} \to \mathfrak{g}^{\mathrm{ad}}$. Then $\mathfrak{g}_A \subset \ker \mathrm{Ad}_C$ and $\ker \mathrm{Ad}_C \cap \mathfrak{g}_C = 0$. Hence $\mathfrak{g}_A \cap \mathfrak{g}_C = 0$, thus $\mathfrak{g} = \mathfrak{g}_A \oplus \mathfrak{g}_C$. By Lemma 4.51 the $\mathrm{d}\alpha^\vee(1)$'s are linearly dependent. Because of the decomposition of $\mathfrak{g}$, the $\mathrm{d}\alpha^\vee(1) : \alpha \in \Delta_A$ are linearly dependent or the $\mathrm{d}\alpha^\vee(1) : \alpha \in \Delta_C$ are linearly dependent. Since p is very good for G_C, the $\mathrm{d}\alpha^\vee(1) : \alpha \in \Delta_C$ are linearly independent. So the $\mathrm{d}\alpha^\vee(1) : \alpha \in \Delta_A$ are linearly dependent. Therefore, we can apply Proposition 4.62 with $H = G$ and $G = G_A$. □

Theorem 4.66. *If G is $\mathbb{F}$-split, then the following are equivalent:*

1. *The number of nilpotent orbits is finite.*

2. *All the nilpotent orbits are separable.*

3. *The regular nilpotent orbit is separable.*

4. *p is good and $p \not| \kappa_v(G)\rho_v(G)$.*

Proof. (2) implies (1) by [McN04, Theorem 40]. (1) implies (4) by Theorem 4.65, Theorem 4.26 and Proposition 4.6. (4) implies (2) by Theorem 4.52. By the proof of Theorem 4.52, not (4) implies not (3). □

Chapter 5

KST-Conjecture for GL_N[1]

Throughout this chapter $\mathcal{B} = \mathcal{B}_e$ denotes the extended building of G. The projection of $x \in \mathcal{B}_e$ to the reduced building will be denoted by $[x] \in \mathcal{B}_r$.

In this chapter we study the following conjecture due to J.-L. Kim, S.W. Shin and N. Templier:

Conjecture 5.1 ([KST16])**.** *Let γ be a regular semisimple element of G. Then for every $\epsilon > 0$ there exists d such that for all square-integrable representations π with unitary central character and* $\deg(\pi) > d$*:*

$$\frac{|\theta_\pi(\gamma)|}{\deg(\pi)} \leq \epsilon.$$

In other words

$$\frac{|\theta_\pi(\gamma)|}{\deg(\pi)} \to 0 \textit{ as } \deg(\pi) \to \infty.$$

They prove this conjecture in [KST16] under the following assumptions:

1. $\mathbb{F}$ is a p-adic field with large enough residual characteristic.
2. π is a tame supercuspidal representation as constructed by J.-K. Yu in [Yu01].
3. $\gamma \in G_{0+} = \bigcup_{x \in \mathcal{B}} G_{x,0+}$.

Here we want to prove the conjecture for the group $GL_N(\mathbb{F})$. In our set-up $\mathbb{F}$ is a non-Archimedean local field and $\gamma \in G_{0+}$ is tamely ramified, i.e., γ is contained in a torus that splits over a tamely ramified field extension of $\mathbb{F}$. In the same way as the construction of J.-K. Yu is used by Kim et al. to prove the conjecture, we will use the construction of supercuspidal representations of $GL_N(\mathbb{F})$ by Bushnell and Kutzko. We will prove two versions of the conjecture for $GL_N(\mathbb{F})$. The first version is with some restrictions on π, specified later. This version is enough to prove the conjecture for $GL_k(\mathbb{F})$, with k prime. The second version is the conjecture for $GL_{kl}(\mathbb{F})$, with k, l prime and $kl > 8$.

[1] The author would like to thank Anne-Marie Aubert for suggesting the topic of this chapter to him and the many useful discussions. The author would also like to thank Ju-Lee Kim for explaining some details of [KST16]. The author learned the basics about strata and supercuspical representations of GL_N during a two month stay at UPMC(Paris VI), partially funded by an Erasmus+Staf Training beurs.

The structure of the argument is the same as in [KST16]. Their argument goes roughly as follows:
Let x be a point in the building of G and J a certain subgroup of $G_{[x]}$. Let ρ be a finite dimensional representation of J as in the construction of J.-K. Yu. Let $\pi = \text{c-Ind}_J^G \rho$ be the irreducible supercuspidal representation of G. If $\deg(\pi)$ is high enough with respect to γ, then

$$\theta_\pi(\gamma) = \operatorname{tr}(\pi(\gamma), V_\pi^{G_{y,r+}}),$$

where $y \in \mathcal{B}_e$ is a point in the apartment of the torus containing γ and $r = l(\pi)$, the level of the representation. We may assume that $G_{y,r+} \subset G_x$. Applying Mackey's theorem we get

$$\theta_\pi(\gamma) = \sum_{g \in G_x \backslash G/J} \operatorname{tr}\left(\pi(\gamma)| \left(\operatorname{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho\right)^{G_{y,r+}}\right).$$

The number of double cosets $G_x g J$ such that $\left(\operatorname{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho\right)^{G_{y,r+}} \neq 0$ is estimated by proving it is contained in two other subsets of G. The first set has to do with the fact that π is supercuspidal. This is already enough to show that the number of double cosets is finite. For the second set the argument is more involved. Since $\left(\operatorname{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho\right)^{G_{y,r+}} \neq 0$, it contains a linear representation of $G_{y,r}$, which is a minimal K-type. Then one shows that g intertwines this linear representation with a particular linear representation of $G_{x,r}$ used in the construction of ρ. By an intertwining theory of (good) minimal K-types, g is then contained in a particular compact modulo center subset of G. Taking the intersection of those two sets, we get the estimate of the number of double cosets $G_x g J$ such that $\left(\operatorname{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho\right)^{G_{y,r+}} \neq 0$.
Now we estimate each term in the sum: $\left|\operatorname{tr}\left(\pi(\gamma)| \left(\operatorname{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho\right)^{G_{y,r+}}\right)\right|$ is bounded by

$$|\{(G_x \cap {}^gJ)k : k \in G_x,\ k\gamma k^{-1} \in (G_x \cap {}^gJ)\}| \dim \rho.$$

The first factor of this product can be bounded independently of g. Combining this with the number of relevant double cosets we get an upper bound for $\theta_\pi(\gamma)$. This upper bound turns out to be small enough to prove the conjecture.

To adjust the proof of [KST16] to our case, we replace the minimal K-types by the simple strata of $GL_N(\mathbb{F})$. We also give a characteristic-free proof for an upper bound of $|\{(G_x \cap {}^gJ)g : g \in G_x,\ g\gamma g^{-1} \in (G_x \cap {}^gJ)\}|$. Our proof uses the Weyl integration formula instead of the exponential map used in [KST16].

5.1 Degree of a representation

In this section we state the definition of the degree of an irreducible representation. Before we can define this notion, we need to introduce smooth dual representations and central characters. This section is based on [Ren10] and [Car85].

Let (π, V) be a smooth representation of G. Define the dual representation $(\pi^\vee, V^\vee)$ by $V^\vee := \{\text{linear functions } f : V \to \mathbb{C}\}$ and $(\pi^\vee(g)f)(v) := f(\pi(g^{-1})v)$. Since $(\pi^\vee, V^\vee)$

is not necessarily smooth, we take the so called smoothening of $V^\vee$:

$$\begin{aligned}\tilde{V} &:= \bigcup_{K\subset G} (V^\vee)^K \\ &= \{f : V \to \mathbb{C} \mid \text{there exists a compact open } K \subset G \text{ with } f(\pi(k)v) = f(v), \text{ for all } k \in K\}.\end{aligned}$$

We define $\tilde{\pi}$ to be the restriction of $\pi^\vee$ on $\tilde{V}$. The representation $(\tilde{\pi}, \tilde{V})$ is called the smooth dual of (π, V). For $v \in V$ and $\tilde{v} \in \tilde{V}$, define the matrix coefficient $\phi_{v,\tilde{v}} : G \to \mathbb{C}$ by $g \mapsto \langle v, \tilde{\pi}(g)\tilde{v}\rangle$.

A representation is called *pre-unitary* if there exists a G-invariant Hermitian inner product on V.

A representation (π, V) is called *square integrable modulo center* if:

- The center Z_G of G acts on V via the unitary character χ.
- Every matrix coefficient is square integrable modulo center, i.e., for every $v, \tilde{v}$:

$$\int_{G/Z_G} |\langle v, \pi(g)\tilde{v}\rangle|^2 \mathrm{d}g^* < \infty.$$

A smooth representation π is called *essentially square integrable modulo center* if there exists a smooth character $\omega : G \to \mathbb{C}^\times$ such that $\omega\pi$ is square integrable modulo center.

The *degree* of an essentially square integrable modulo center representation is the number $\deg(\pi)$ such that

$$\int_{G/Z_G} \langle \pi(g)v_1, \tilde{u}_1\rangle \langle \pi(g^{-1})v_2, \tilde{u}_2\rangle \deg(\pi) dg^* = \langle v_1, \tilde{u}_2\rangle \langle v_2, \tilde{u}_1\rangle,$$

for all $v_1, v_2 \in V, \tilde{u}_1, \tilde{u}_2 \in \tilde{V}$. By definition, the degree of π depends on the Haar measure on G/Z_G. In the KST-conjecture we fix a Haar measure. For every smooth $\chi : G \to \mathbb{C}^\times$ and every essentially square integrable modulo center representation π, $\deg(\pi) = \deg(\chi\pi)$. See [Ren10, Lemma IV.3.3] for a proof of the existence of such a number.

Lemma 5.2. *Let J be a compact modulo center subgroup of G containing Z_G, the center of G. Let ρ be a finite dimensional representation of J, such that $\pi = c\text{-}Ind_J^G(\rho)$ is an essentially square integrable representation of G. Then*

$$\mu_{G/Z_G}(J/Z_G)\deg(\pi) = \dim(\rho).$$

Proof. See [BH96, Theorem A.14]. □

5.2 Bushnell-Kutzko construction

In this section we recall some parts of the Bushnell-Kutzko construction of supercuspidal representations. For a detailed account of their theory one could read [BK93]. However, for the users of the BK-construction and for the purpose of this chapter, the author recommends the notes of W. Conley [Con09] and [BK93, §2.5]. See [BK93] or [Con09] for the definitions of the notions/symbols undefined in this section.

Let $G = GL_N(\mathbb{F})$. Define $A = \mathrm{End}_{\mathbb{F}}(\mathbb{F}^N)$. Let $\beta \in A$. Assume $\mathbb{E} = \mathbb{F}[\beta]$ is a field. Define $B_\beta = \mathrm{End}_{\mathbb{E}}(\mathbb{F}^N) = Z_A(\beta)$. For an order $\mathfrak{A}$, define

$$\begin{aligned}\mathfrak{B}_\beta &:= \mathfrak{A} \cap B_\beta,\\ \mathfrak{N}_k &:= \{x \in \mathfrak{A} : [x,\beta] \in \mathfrak{P}^k\},\\ k_0(\beta,\mathfrak{A}) &:= \max\{k \in \mathbb{Z} \mid \mathfrak{N}_k \not\subset \mathfrak{B} + \mathfrak{P}\}.\end{aligned}$$

All the irreducible supercuspidal representations of depth greater than 0 can be constructed in the following way (up to isomorphism):

Let $\mathfrak{A}$ be a principal order.
Let $[\mathfrak{A}, n, 0, \beta_0]$ be a simple stratum with $e(\mathfrak{A}) = e(\mathbb{F}[\beta_0] : \mathbb{F})$.
Let $[\mathfrak{A}, n, r_i, \beta_i]$, $0 \le i \le l$, be a defining sequence for this stratum. Define $\beta = \beta_0$.

For $k = 0, 1$, define the groups

$$\begin{aligned}H^k(\beta,\mathfrak{A}) &= U^k(\mathfrak{B}_{\beta_0})U^{\lfloor \frac{r_1}{2}\rfloor+1}(\mathfrak{B}_{\beta_1})\cdots U^{\lfloor \frac{r_l}{2}\rfloor+1}(\mathfrak{B}_{\beta_l})U^{\lfloor \frac{n}{2}\rfloor+1}(\mathfrak{A}),\\ J^k(\beta,\mathfrak{A}) &= U^k(\mathfrak{B}_{\beta_0})U^{\lfloor \frac{r_1+1}{2}\rfloor}(\mathfrak{B}_{\beta_1})\cdots U^{\lfloor \frac{r_l+1}{2}\rfloor}(\mathfrak{B}_{\beta_l})U^{\lfloor \frac{n}{2}\rfloor+1}(\mathfrak{A}).\end{aligned}$$

Let $\mathcal{C}(\mathfrak{A}, 0, \beta)$ be the set of simple characters of $H^1(\beta,\mathfrak{A})$. Then according to [Con09, Corollary 2.2.3(b)], $\theta = \psi_\beta$ on $U^n(\mathfrak{A})$ for all $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. Take $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. Let η be the unique irreducible representation of $J^1(\beta,\mathfrak{A})$ which contains θ when viewed as $H^1(\beta,\mathfrak{A})$-representation. Then η is, as $H^1(\beta,\mathfrak{A})$-representation, a direct sum of copies of θ. Let κ be a β-extension of η. Then κ is by definition an irreducible $J^0(\beta,\mathfrak{A})$-representation whose restriction to $J^1(\beta,\mathfrak{A})$ is equal to η. Let $\lambda = \kappa \otimes \sigma$, with σ an inflation of a representation of $J^0(\beta,\mathfrak{A})/J^1(\beta,\mathfrak{A})$ such that $(J^0(\beta,\mathfrak{A}), \lambda)$ is a simple type. Let ρ be an extension of λ to $J := \mathbb{F}[\beta]^\times J^0(\beta,\mathfrak{A})$, then $\pi := \text{c-Ind}_J^G(\rho)$ is an irreducible supercuspidal representation of G.

In the proof of the KST-conjecture for $GL_N(\mathbb{F})$ we will use the following properties of the BK-construction:

- $U^{\lfloor \frac{n}{2}\rfloor+1}(\mathfrak{A}) \subset J$.
- Λ restricted to $U^n(\mathfrak{A})$ is equal to a finite direct sum of copies of ψ_{β_l}.

For most of this chapter we fix the choices we made to construct the supercuspidal representation $\pi = \text{c-Ind}_J^G(\Lambda)$ out of the simple stratum $[\mathfrak{A}, n, 0, \beta]$.

In order to go back and forth between the general Bruhat-Tits notation and the orders, we take $x \in \mathcal{B}$ such that $G_x = \mathfrak{A}^\times$.

We fix a compact regular semisimple element $\gamma \in G_{0+}$ throughout this chapter.

To get a sufficiently low estimate of the $\theta_\pi(\gamma)$ we also need to assume the following:

Hypothesis 1. *The element $\beta_l \in GL_N(\mathbb{F})$, as in the defining sequence, satisfies*

$$J \subset Z_G(\beta_l)U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}).$$

We say that an irreducible supercuspidal representation π of $GL_N(\mathbb{F})$ satisfies Hypothesis 1, if π is isomorphic to a representation constructed via a defining sequence satisfying Hypothesis 1.
This hypothesis is at least satisfied in the following cases:

- $\mathbb{F}[\beta]$ is a tamely ramified field extension of $\mathbb{F}$.
- β is a minimal element of $\mathbb{F}$. Hence the hypothesis is satisfied by all supercuspidal representations of $GL_k(\mathbb{F})$, with k prime.

Every group $GL_N(\mathbb{F})$ has, for every n, a minimal element β of $\mathbb{F}$ such that $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum. Thus there are infinitely many supercuspidal representations of $GL_N(\mathbb{F})$ that can be constructed via simple strata satisfying the hypothesis. However, as the example in §5.3 tells us, the hypothesis is not always satisfied. We only use this hypothesis in the end of the proof. We will clearly indicate in which subsections we use this hypothesis.

Since γ is compact, the absolute value of $\theta_\pi(\gamma)$ does not change if we twist π with a character $\chi : \mathbb{F}^\times \to \mathbb{C}^\times$. Hence we may and will assume that $l(\pi)$, the level of π, is the lowest among the set $\{l(\chi\pi) : \chi \text{ character of } \mathbb{F}\}$. A consequence of this assumption is that $\mathbb{F}[\beta_l]$ is not equal to $\mathbb{F}$ (see [BH06, Theorem 13.3]).

5.3 An example of a defining sequence

This section can be skipped by the reader only interested in proofs of the KST-conjecture. It is only included in this chapter to show that Hypothesis 1 is not fulfilled by all defining sequences. This example is based on a suggestion by Daniel Skodlerack and David Helm sent by Anne-Marie Aubert to the author by email.

We will construct a defining sequence by the method of Bushnell and Kutzko from the second and last stratum in the defining sequence to the first stratum (see [BK93, Proposition 2.2.3]).
We take $\mathbb{F} = \mathbb{Q}_2$ and $G = GL_4(\mathbb{Q}_2)$.
The order and its ideals are chosen as follows:

$$\mathfrak{A} = \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}, \qquad \mathfrak{P} = \begin{pmatrix} 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & 2\mathbb{Z}_2 \end{pmatrix},$$

$$\mathfrak{P}^2 = \begin{pmatrix} 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 4\mathbb{Z}_2 \\ 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \end{pmatrix}, \qquad \mathfrak{P}^3 = \begin{pmatrix} 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 4\mathbb{Z}_2 & 4\mathbb{Z}_2 \\ 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 4\mathbb{Z}_2 \\ 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \end{pmatrix}.$$

We take $\beta = \frac{1}{4}\sqrt{2}$ and $\alpha = \frac{1}{4}(2 + \sqrt[4]{2}^3)$. We embed $\mathbb{F}[\sqrt[4]{2}]$ into $A := M_4(\mathbb{Q}_2)$ as follows:

$$\sqrt[4]{2} \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this way $\mathbb{F}[\sqrt[4]{2}]^\times \subset \mathcal{K}(\mathfrak{A})$. The real numbers β and α correspond with:

$$\beta \leftrightarrow \frac{1}{4}\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \alpha \leftrightarrow \frac{1}{4}\begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

So the centralizer of β is:

$$B := B_\beta = \left\{ b' = \begin{pmatrix} a_1 & a_2 & 2b_1 & 2b_2 \\ a_3 & a_4 & 2b_3 & 2b_4 \\ b_1 & b_2 & a_1 & a_2 \\ b_3 & b_4 & a_3 & a_4 \end{pmatrix} \mid a_i, b_i \in \mathbb{Q}_2 \right\}.$$

When the vector space is viewed over $\mathbb{Q}_2(\sqrt{2}) = \mathbb{Q}_2[\beta]$, we get the following correspondence:

$$\begin{pmatrix} a_1 & a_2 & 2b_1 & 2b_2 \\ a_3 & a_4 & 2b_3 & 2b_4 \\ b_1 & b_2 & a_1 & a_2 \\ b_3 & b_4 & a_3 & a_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} a_1 + \sqrt{2}b_1 & a_2 + \sqrt{2}b_2 \\ a_3 + \sqrt{2}b_3 & a_4 + \sqrt{2}b_4 \end{pmatrix} \in M_2(\mathbb{Q}_2(\sqrt{2})).$$

Then $[\mathfrak{A}, 6, 5, \beta]$ and $[\mathfrak{B}, 5, 4, \alpha]$ are simple strata (both are minimal elements of $\mathbb{Q}_2$ and $\mathbb{Q}_2[\sqrt{2}]$, respectively). Now we need to find a $b \in GL_4(\mathbb{Q}_2)$ such that $s(b) = \alpha$, where s is a tame corestriction. We define s with respect to the following additive characters of $\mathbb{Q}_2$ and $\mathbb{Q}_2[\sqrt{2}]$ (see [BK93, Proposition 1.3.4]):

$$\begin{aligned} \Psi_{\mathbb{Q}_2}(x) &:= \exp\left(\frac{x}{2}\right), \\ \Psi_{\mathbb{Q}_2[\sqrt{2}]}(x + y\sqrt{2}) &:= \exp\left(\frac{x}{2}\right) = \Psi_{\mathbb{Q}_2}(x) \end{aligned}$$

for $x, y \in \mathbb{Q}_2$. Thus, for $b \in GL_4(\mathbb{Q}_2)$, $s(b)$ is the element in B such that, for all $b' \in B$,

$$\Psi_{\mathbb{Q}_2} \circ \mathrm{Tr}_A(bb') = \Psi_{\mathbb{Q}_2[\sqrt{2}]} \circ \mathrm{Tr}_B(s(b)b').$$

To find a b with $s(b) = \alpha$, we first calculate $\mathrm{Tr}_B(\alpha b')$.

$$\begin{aligned} \mathrm{Tr}_B(\alpha b') &= \mathrm{Tr}_B\left(\frac{1}{4}\begin{pmatrix} 2 & 1 \\ \sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} a_1 + \sqrt{2}b_1 & a_2 + \sqrt{2}b_2 \\ a_3 + \sqrt{2}b_3 & a_4 + \sqrt{2}b_4 \end{pmatrix}\right) \\ &= \frac{2}{4}(a_1 + a_3 + a_4 + b_2) + \frac{\sqrt{2}}{4}(2b_1 + b_3 + a_2 + 2b_4). \end{aligned}$$

Therefore

$$\Psi_{\mathbb{Q}_2[\sqrt{2}]} \circ \mathrm{Tr}_B(\alpha b') = \Psi_{\mathbb{Q}_2}\left(\frac{2}{4}(a_1 + a_3 + a_4 + b_2)\right).$$

So, by [BK93, Proposition 1.3.4], we need to find a $b \in GL_4(\mathbb{Q}_2)$ such that $\Psi_{\mathbb{Q}_2}(\mathrm{Tr}_A(bb')) = \Psi_{\mathbb{Q}_2}\left(\frac{2}{4}(a_1 + a_3 + a_4 + b_2)\right)$. We take

$$b = \frac{1}{4}\begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

According to [BK93, Proposition 2.2.3], then $[\mathfrak{A}, 6, 4, \beta + b]$ is a simple stratum.
To demonstrate this directly, we do some calculations. Define $X := 4(\beta + b)$. Then

$$X = \begin{pmatrix} 2 & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \; \frac{1}{2}X^2 = \begin{pmatrix} 3 & 4 & 2 & 2 \\ 1 & 3 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix}, \; \frac{1}{2}X^3 = \begin{pmatrix} 10 & 16 & 6 & 8 \\ 4 & 10 & 2 & 6 \\ 3 & 4 & 2 & 2 \\ 4 & 7 & 2 & 4 \end{pmatrix},$$

$$\mathfrak{B}_{\beta+b} = \left\langle I_4, X, \frac{1}{2}X^2, \frac{1}{2}X^3 \right\rangle_{\mathbb{Z}_2}.$$

Therefore, $\mathfrak{B} + \mathfrak{P} = aI_4 + \mathfrak{P}$. We will now calculate $k_0(\beta + b, \mathfrak{A})$.

$$\left[X, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{P}^3.$$

Thus $k_0(\beta + b, \mathfrak{A}) \geq -5$. Now we show that $\mathfrak{N}_{-4}(\beta + b, \mathfrak{A}) \subset \mathfrak{B} + \mathfrak{P}$.
Let $a, f, k, p, e, j, o, i, n, m \in \mathbb{Z}_2$ and $b, c, d, g, h, l \in 2\mathbb{Z}_2$.

$$\left[X, \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}\right]$$
$$= \begin{pmatrix} -c-d+2e+2i & -2a-d+2f+2j & -2a+2c+2g+2k & -2b+2d+2h+2l \\ -g-h+2m & -2e-h+2n & -2e+2g+2o & -2f+2h+2p \\ a-2i-k-l & b-2i-2j-l & c-2i & d-2j \\ a+e-2m-o-p & b+f-2m-2n-p & c+g-2m & d+h-2n \end{pmatrix}.$$

We conclude that

$$\left[X, \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}\right] \in 2\mathfrak{A} \Rightarrow \begin{cases} a - k \equiv 0 & (13) \\ f - p \equiv 0 & (42) \\ a + e \equiv p + o & (41) \\ e \equiv o & (23) \end{cases}.$$

where the equivalence is modulo 2 and between parentheses the place of the relation which implies the equivalence relation. Thus $f \equiv p \equiv a \equiv k \mod 2$. Therefore, $\mathfrak{N}_{-4}(\beta+b,\mathfrak{A}) \subset \mathfrak{B}+\mathfrak{P}$. Hence $k_0(\beta+b,\mathfrak{A}) = -5$.

The characteristic polynomial of $1+\frac{1}{2}X^2$ is $x^4-12x^3+40x^2-50x+22$. This is an Eisenstein polynomial, hence irreducible. Thus $\mathbb{F}[X]$ is a field.
Since

$$(1+\frac{1}{2}X^2)\begin{pmatrix}0&0&0&2\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}^{-1} = \begin{pmatrix}1&4&4&2\\1&1&4&0\\0&1&1&2\\1&1&2&1\end{pmatrix} \in \mathfrak{A}^\times,$$

$(1+\frac{1}{2}X^2)L_i = L_{i+1}$. The lattices L_i are also $\mathcal{O}_{\mathbb{E}}$-lattices, because $\mathbb{E}:\mathbb{F}$ is totally ramified and $1+\frac{1}{2}X^2$ is a uniformizer of $\mathbb{E}$. Hence $\mathbb{F}[X]^\times \subset \mathcal{K}(\mathfrak{A})$. Thus $[\mathfrak{A},6,4,b+\beta]$ is a simple stratum and its defining sequence is $[\mathfrak{A},6,4,b+\beta]$, $[\mathfrak{A},6,5,\beta]$.
We claim $(1+\frac{1}{2}X^2)\cdot(\sqrt[4]{2})^{-1} \notin (Z_G(\beta)\cap\mathcal{K}(\mathfrak{A}))U^3(\mathfrak{A})$. Otherwise, let $Y \in Z_G(\beta)\cap\mathcal{K}(\mathfrak{A})$ such that $(1+\frac{1}{2}X^2)\cdot(\sqrt[4]{2})^{-1} \in YU^3(\mathfrak{A})$. Then $Y \in \mathfrak{A}^\times$. Hence, for some $a_i, b_i \in \mathcal{O}$,

$$\begin{pmatrix}1&4&4&2\\1&1&4&0\\0&1&1&2\\1&1&2&1\end{pmatrix} \equiv \begin{pmatrix}a_1&a_2&2b_1&2b_2\\a_3&a_4&2b_3&2b_4\\b_1&b_2&a_1&a_2\\b_3&b_4&a_3&a_4\end{pmatrix} \mod \mathfrak{P}^3.$$

Then $a_3 \equiv 1 \equiv 2 \mod 2$ is a contradiction.
Thus $1+\frac{1}{2}X^2 \in \mathbb{F}[\beta+b]^\times$ and $1+\frac{1}{2}X^2 \notin Z_G(\beta)U^3(\mathfrak{A})$. Hence $J \not\subset Z_G(\beta)U^3(\mathfrak{A})$.

5.4 Combinatorial estimates

This section consists of a collection of estimates on the number of elements in cosets and double cosets.

5.4.1 Conjugated into compact subgroup

We will first prove a theorem and a lemma for a general reductive p-adic group, not only for $GL_N(\mathbb{F})$. Then we will calculate some estimates specific for $GL_N(\mathbb{F})$. In this subsection we will give an upper bound for

$$|\{(G_x \cap {}^gJ)k : k \in G_x,\ k\gamma k^{-1} \in (G_x \cap {}^gJ)\}|.$$

For a regular semisimple element γ, define T^γ to be the torus containing γ, i.e. $T^\gamma = Z_G^0(\gamma)$.

Theorem 5.3. *Let G be a reductive p-adic group. Let H be a compact modulo center open subgroup of G. Let $s>0$ and $x \in \mathcal{B}$ be such that $G_{x,s} < H < G_{[x]}$. Let $z \in \mathcal{B}$. If γ is a regular semisimple element such that $\gamma \in H \cap G_z$, then*

$$T^\gamma_{(\mathrm{sd}(\gamma)\mathrm{ht}(\Phi)+s)+} \subset H \cap G_z.$$

This theorem is a slightly more general version of [KST16, Lemma 3.5].

Proof. We consider first the case that γ and G are $\mathbb{F}$-split. Take $u \in U^+$ such that $y = ux \in \mathbb{A}(T^\gamma)$. Define $z' = uz$. Now $\gamma \in H \cap G_z$ if and only if $u\gamma u^{-1} \in uHu^{-1} \cap G_{z'}$. Since G_y is the maximal compact subgroup contained in $G_{[y]}$ and $uHu^{-1} \subset G_{[y]}$, $uHu^{-1} \cap G_{z'} \subset G_y$. Since $\gamma \in G_y$ and $u\gamma u^{-1} \in uHu^{-1} \cap G_{z'}$, also $u\gamma u^{-1}\gamma^{-1} \in G_y$.

Fix, for all $\alpha \in \Phi^+$, a pinning of the root subgroups U_α such that $G_y \cap U_\alpha = u_\alpha(\mathcal{O})$. Fix an order on the roots of G and T. For $u \in U^+$, define $\lambda_\alpha \in \mathbb{F}$ such that $u = \prod_{\alpha\in\Phi^+} u_\alpha(\lambda_\alpha)$. Let $\gamma_+ \in T^\gamma_{(\mathrm{sd}(\gamma)\mathrm{ht}(\Phi)+s)+}$. Let $\alpha \in \Phi^+$.
In the notation of [MS12, Lemma 4.3]: The λ_α corresponding to $[u,\gamma]_\alpha$ becomes a sum of terms of the form $c_i(\beta_i(\gamma)-1)\lambda_{\beta_1}\cdots\lambda_{\beta_j}$, with $\beta_k \in \Phi$, $\sum_{k=1}^j \beta_k = \alpha$ and $c_i \in \mathcal{O}^\times$. The commutator $[u,\gamma\gamma_+]_\alpha$ is equal to the same sum of terms with $\beta_i(\gamma\gamma_+)-1$ instead of $\beta_i(\gamma)-1$. By [MS12, Proposition 4.2(c)], since γ fixes $[x]$, $\nu(\lambda_{\beta_k}) \geq -\mathrm{ht}(\beta_k)\mathrm{sd}(\gamma)$. Therefore,

$$\begin{aligned}
\nu\left(c_i(\beta_i(\gamma)-1)\lambda_{\beta_1}\cdots\lambda_{\beta_j} - c_i(\beta_i(\gamma\gamma_+)-1)\lambda_{\beta_1}\cdots\lambda_{\beta_j}\right)& \\
&= \nu\left(\beta_i(\gamma)-\beta_i(\gamma\gamma_+)\right) + \sum_{i=1}^k \nu(\lambda_{\beta_k}) \\
&\geq s + \mathrm{sd}(\gamma)\mathrm{ht}(\Phi) - \sum_{i=1}^k \mathrm{ht}(\beta_k)\mathrm{sd}(\gamma) \\
&= s + \mathrm{sd}(\gamma)(\mathrm{ht}(\Phi)-\mathrm{ht}(\alpha)) \\
&\geq s.
\end{aligned}$$

Thus for all $\alpha \in \Phi^+$:

$$(u\gamma\gamma_+u^{-1}\gamma_+^{-1}\gamma^{-1})_\alpha \equiv (u\gamma u^{-1}\gamma^{-1})_\alpha \mod \varpi^s.$$

Therefore, modulo $G_{y,s}$:

$$\begin{aligned}
u\gamma\gamma_+u^{-1} &= u\gamma\gamma_+u^{-1}\gamma_+^{-1}\gamma^{-1}\gamma\gamma_+ \\
&\equiv u\gamma u^{-1}\gamma^{-1}\gamma\gamma_+ \\
&= u\gamma u^{-1}\gamma_+ \equiv u\gamma u^{-1}.
\end{aligned}$$

Therefore, $u\gamma\gamma_+u^{-1} \in uHu^{-1}$. Thus $\gamma\gamma_+ \in H$. By [MS12, Lemma 4.3], also $\gamma\gamma_+ \in G_z$. Hence $T^\gamma_{(\mathrm{sd}(\gamma)\mathrm{ht}(\Phi)+s)+} \subset H \cap G_z$, which proves the split case.

Let $\mathbb{E}$ be a finite field extension of $\mathbb{F}$ such that γ and G are $\mathbb{E}$-split. Define $H(\mathbb{E}) := HG(\mathbb{E})_{x,s}$. Since $H \subset G_{[x]} \subset G(\mathbb{E})_{[x]}$, it is a subgroup with $G(\mathbb{E})_{x,s} < H(\mathbb{E}) < G(\mathbb{E})_{[x]}$. Also $H(\mathbb{E}) \cap G = H$, because $G(\mathbb{E})_{x,s} \cap G(\mathbb{F}) = G(\mathbb{F})_{x,s}$. We just showed that $T^\gamma(\mathbb{E})_{(\mathrm{sd}(\gamma)\mathrm{ht}(\Phi)+s)+} \subset H(\mathbb{E}) \cap G_z(\mathbb{E})$. Thus

$$T^\gamma_{(\mathrm{sd}(\gamma)\mathrm{ht}(\Phi)+s)+} \subset (H(\mathbb{E}) \cap G(\mathbb{F})) \cap (G_z(\mathbb{E}) \cap G(\mathbb{F})) = H \cap G_z. \qquad \square$$

Let $Z_{G,0}$ be the maximal compact subgroup of Z_G, the center of G. Define, for $\gamma \in G$,

$$\psi_\gamma : G \to G \text{ by } g \mapsto g\gamma g^{-1}.$$

Lemma 5.4. *Let G be a reductive p-adic group and $x \in \mathcal{B}_e(G)$. Let $H < G_x$ be an open compact subgroup containing $Z_{G,0}$. Let $r \geq 1$ and $s \geq 0$ be such that for all regular semisimple elements $\eta \in G$: if $\eta \in H$, then $T^{\eta}_{(r \cdot \mathrm{sd}(\eta)+s)+} \subset H$. Assume γ is a regular semisimple element in H. Define $k := r \cdot \mathrm{sd}(\gamma) + s$ and define W to be the order of the Weyl group of T^{γ}. Then*

$$[\psi_{\gamma}^{-1}(H) \cap G_x : H] \leq W|D(\gamma)|^{-1}[T_0 : Z_{G,0}T_{k+}].$$

Proof. We are going to estimate the index by estimating the volume of $\psi_{\gamma}^{-1}(H) \cap G_x$ in terms of the volume of H in G. Take on G and on $T := T^{\gamma}$ the measures μ_G and μ_T such that $\mu_G(G_x) = \mu_T(G_x \cap T) = 1$. Define the measure $\mu_{G/T}$ as usual by: $\mu_G = \mu_T \mu_{G/T}$. Then

$$\begin{aligned}
\mu_G(\psi_{\gamma}^{-1}(H) \cap G_x) &= \int_G 1_{G_x}(\sigma) 1_H(\sigma\gamma\sigma^{-1}) \mathrm{d}\sigma \\
&= \int_{G/T} \int_T 1_{G_x}(\sigma t) 1_H(\sigma t \gamma t^{-1} \sigma^{-1}) \mathrm{d}t \mathrm{d}\sigma \\
&= \int_{G/T} 1_H(\sigma\gamma\sigma^{-1}) \int_T 1_{G_x}(\sigma t) \mathrm{d}t \mathrm{d}\sigma \\
&\leq \int_{G/T} 1_H(\sigma\gamma\sigma^{-1}) \mathrm{d}\sigma.
\end{aligned}$$

We claim that, for $\sigma \in G$ and for $\gamma_0, \gamma_1 \in \gamma T_k$,

$$\sigma\gamma_0\sigma^{-1} \in H \Leftrightarrow \sigma\gamma_1\sigma^{-1} \in H.$$

Indeed if $\eta = \sigma\gamma_0\sigma^{-1} \in H$, then $T^{\eta}_{(r \cdot \mathrm{sd}(\eta)+s)+} \subset H$ follows from the assumption on H. Therefore,

$$\sigma\gamma_0 T^{\gamma}_{(r \cdot \mathrm{sd}(\gamma)+s)+}\sigma^{-1} = \eta T^{\eta}_{(r \cdot \mathrm{sd}(\eta)+s)+} \subset H.$$

Thus $\eta \mapsto \mu_G(\psi_{\eta}^{-1}(H) \cap G_x)$ is constant on the set γT_{k+}. Since $Z_{G,0} \subset H$, this function is even constant on $\gamma Z_{G,0} T_{k+}$. Thus

$$\begin{aligned}
\mu_T(Z_{G,0}T_{k+})\mu_G(\psi_{\gamma}^{-1}(H) \cap G_x) &\leq \int_{\gamma Z_{G,0}T_{k+}} \int_{G/T} 1_H(\sigma t \sigma^{-1}) \mathrm{d}\sigma \mathrm{d}t \\
&= |D(\gamma)|^{-1} \int_{\gamma Z_{G,0}T_{k+}} |D(t)| \int_{G/T} 1_H(\sigma t \sigma^{-1}) \mathrm{d}\sigma \mathrm{d}t \\
&\leq |D(\gamma)|^{-1} \int_T |D(t)| \int_{G/T} 1_H(\sigma t \sigma^{-1}) \mathrm{d}\sigma \mathrm{d}t \\
&= |D(\gamma)|^{-1} W \int_{{}^G T} 1_H(x) \mathrm{d}x \\
&\leq |D(\gamma)|^{-1} W \mu_G(H),
\end{aligned}$$

where the last equality is due to the Weyl integration formula. Thus

$$\frac{\mu_G(\psi_{\gamma}^{-1}(H) \cap G_x)}{\mu_G(H)} \leq \frac{|D(\gamma)|^{-1} W}{\mu_T(Z_{G,0}T_{k+})}.$$

Now we will estimate $\mu_T(Z_{G,0}T_{k+})$. Let T_0 be the maximal compact subgroup of T. Since $G_x \cap T \subset T_0$ and $\mu_T(G_x \cap T) = 1$, we have $\mu_T(T_0) \geq 1$. Thus

$$\frac{1}{\mu_T(Z_{G,0}T_{k+})} = \frac{[T_0 : Z_{G,0}T_{k+}]}{\mu_T(T_0)} \leq [T_0 : Z_{G,0}T_{k+}].$$

Therefore,

$$[\psi_\gamma^{-1}(H) \cap G_x : H] = \frac{\mu_G(\psi_\gamma^{-1}(H) \cap G_x)}{\mu_G(H)} \leq |D(\gamma)|^{-1} W [T_0 : Z_{G,0}T_{k+}]. \qquad \square$$

The proof does not take the $[T_0 : T \cap G_x]$ into account. A closer analysis of this index could lead to a smaller estimate.

Lemma 5.5. *Let $\beta \in GL_N(\mathbb{F})$, $x \in \mathcal{B}$ and $s \in \mathbb{R}_{\geq 0}$, then*

$$[G_{x,0+} : (Z_G(\beta) \cap G_{x,0+})\, G_{x,s}] \geq q^{(\dim G - \dim Z_G(\beta))(\lfloor s \rfloor - 1)}.$$

Proof. Let $n := \lfloor s \rfloor - 1$, then $n < s$. Hence it is enough to show:

$$[G_{x,0+} : (Z_G(\beta) \cap G_{x,0+})\, G_{x,n+}] \geq q^{(\dim G - \dim Z_G(\beta))n}.$$

Take $\mathfrak{g}(\mathcal{O}) \subset \mathfrak{g}$ such that $G_{x,0+} = 1 + \mathfrak{g}(\mathcal{O})$, as subsets of $\mathrm{End}_{\mathbb{F}}(\mathbb{F}^N)$. Let A be an $(N^2 \times N^2)$-matrix such that for $X \in \mathfrak{g}(\mathcal{O})$:

$$1 + X \in Z_G(\beta) \Leftrightarrow AX = 0,$$

with X on the right-hand side viewed as an element of a N^2-dimensional vector space. Let $z := \dim Z_G(\beta)$. Take $X_1, \ldots, X_{N^2}$ an $\mathcal{O}$-basis for $\mathfrak{g}(\mathcal{O})$ such that $X_1, \ldots, X_z$ span the kernel of A. Such a basis exists, since $A = PDQ$ with $P, Q \in GL_N(\mathcal{O})$ and D a diagonal matrix called the Smith normal form of A. Now $AX = 0$ if and only if $D(QX) = 0$. Then

$$1 + \sum_{i=1}^{N^2} c_i X_i \in (Z_G(\beta) \cap G_{x,0+})\, G_{x,n+} \Leftrightarrow \nu(c_i) \geq n \text{ for all } i > z.$$

Thus

$$[G_{x,0+} : (Z_G(\beta) \cap G_{x,0+})\, G_{x,n+}] = q^{(\dim G - \dim Z_G(\beta))n}. \qquad \square$$

For $r \in \mathbb{R}_{\geq 0}$, define the following subgroup of $\mathcal{O}^\times$:

$$\mathcal{O}_r^\times := \{x \in \mathcal{O}^\times \mid \nu(x - 1) \geq r\}.$$

Lemma 5.6. *Let $T \subset GL_N(\mathbb{F})$ be a tamely ramified maximal torus, then, for all $s \in \mathbb{R}_{\geq 0}$,*

$$[T_{0+} : \mathcal{O}_1^\times T_{s+}] \leq q^{(N-1)\lceil s \rceil}.$$

Proof. Let $x \in \mathcal{B}$ such that x is also in the apartment of T. Take $\mathfrak{g}(\mathcal{O})$ to be the $\mathcal{O}$-lattice in $\mathfrak{g}$, such that $1 + \mathfrak{g}(\mathcal{O}) = G_{x,0+}$. Let β be an element in T such that $Z_G(\beta) = T$. Then by the same argument as in Lemma 5.5, there exists a lattice basis $X_1, \ldots, X_{N^2}$ of $\mathfrak{g}(\mathcal{O})$ such that $X_1, \ldots, X_N$ is a basis for $\mathfrak{t}$. For $n \in \mathbb{N}$,

$$1 + \sum_{i=1}^{N^2} c_i X_i \in G_{x,n+} \Leftrightarrow \nu(c_i) \geq n.$$

Since x is in the apartment of T, $G_{x,r} \cap T = T_r$ for all $r > 0$. Thus

$$1 + \sum_{i=1}^{N} c_i X_i \in T_{n+} \Leftrightarrow \nu(c_i) \geq n.$$

Thus $[T_{0+} : T_{n+}] = q^{Nn}$. Since

$$\begin{aligned}
[\mathcal{O}_1^\times T_{n+} : T_{n+}] &= [\mathcal{O}_1^\times : \mathcal{O}_{n+1}^\times] = q^n, \\
[T_{0+} : \mathcal{O}_1^\times T_{n+}] &= \frac{[T_{0+} : T_{n+}]}{[\mathcal{O}_1^\times T_{n+} : T_{n+}]} = q^{(N-1)n}.
\end{aligned}$$

The lemma follows. □

Define $H_g := {}^g J \cap G_x$ with J as in §5.2. Define $s := \frac{\lfloor \frac{n}{2} \rfloor + 1}{e(\mathfrak{A})}$. Then $U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}) = G_{x,s}$. Thus $G_{x,s} \subset J$.

Corollary 5.7. *For all tamely ramified semisimple regular $\gamma \in GL_N(\mathbb{F})$ and $g \in GL_N(\mathbb{F})$, such that $\gamma \in H_g$:*

$$[\psi_\gamma^{-1}(H_g) \cap G_x : H_g] \leq W |D(\gamma)|^{-1} q^{N(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma)+2)} q^{(N-1)s}.$$

Proof. Since $G_{x,s} < J < G_{[x]}$, also $G_{gx,s} < {}^g J < G_{[gx]}$. By Theorem 5.3, for all semisimple regular $\delta \in H_g$, also $T^\delta_{(\mathrm{sd}(\delta)\mathrm{ht}(\Phi)+s)+} \subset H_g$. By definition of J, $Z_G \subset J$. Thus $Z_{G,0} \subset H_g$. Now apply Lemma 5.6 and Lemma 5.4 (with $r = \mathrm{ht}(\Phi)$). □

5.4.2 The double coset estimate

Define $A := M_N(\mathbb{F}) = \mathrm{End}_{\mathbb{F}}(\mathbb{F}^N)$.
For $g \in G$, define

$$\begin{aligned}
P_g &:= \{p \in G : \{g^n p g^{-n} : n \in \mathbb{N}\} \text{ is bounded}\} \text{ and} \\
M_g &:= \{p \in G : \{g^n p g^{-n} : n \in \mathbb{Z}\} \text{ is bounded}\}.
\end{aligned}$$

Define U_g^+ to be the unipotent radical of P_g and U_g^- to be the unipotent radical of the parabolic subgroup opposite to P_g.

Lemma 5.8. *Let $\delta \in A$, such that $\mathbb{E} := \mathbb{F}[\delta]$ is a field. Let $\mathfrak{A}$ be an $\mathcal{O}_{\mathbb{F}}$-order. Let z be the point in the building corresponding to $\mathfrak{A}$. Assume $\mathbb{E}^\times \subset G_{[z]}$. Then $\mathfrak{A} \cap B_\delta$ is an $\mathcal{O}_{\mathbb{E}}$-order. Define $G' := Z_G(\delta) = Aut_{\mathbb{E}}(\mathbb{F}^N)$. Then there exists a maximal $\mathbb{E}$-split torus $T' \subset G'$ such that z lies in the apartment of T' and for all $t' \in T'$:*

$$U^1(\mathfrak{A}) = (U^1(\mathfrak{A}) \cap U_{t'}^-)(U^1(\mathfrak{A}) \cap M_{t'})(U^1(\mathfrak{A}) \cap U_{t'}^+).$$

The group $Z_G(\delta)$ is not necessarily a (standard) Levi subgroup of G.

Proof. We embed the building $\mathcal{B}(G')$ in $\mathcal{B}(G)$ by the map in [BL02, II, Theorem 1.1]. In this way the points of $\mathcal{B}(G')$ are identified with the points $x \in \mathcal{B}(G)$ with $\mathbb{E}^\times \in G_{[x]}$. By choosing an $\mathbb{F}$-basis of $\mathbb{E}$, we view $\mathbb{E}$ as a subalgebra of $M_m(\mathbb{F})$, where $m = [\mathbb{E} : \mathbb{F}]$. Let $l = N/m$.
By conjugating with an appropriate matrix in $GL_N(\mathbb{F})$, we may assume that

$$\mathbb{F}[\delta] = \left\{ \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A \end{pmatrix} : A \in \mathbb{E} \right\},$$

$$G' = \left\{ \begin{pmatrix} A_{11} & \cdots & A_{1l} \\ \vdots & \ddots & \vdots \\ A_{l1} & \cdots & A_{ll} \end{pmatrix} \in GL_N(\mathbb{F}) \mid A_{ij} \in \mathbb{E} \right\} = GL_l(\mathbb{E}).$$

Then we take the diagonal torus as maximal $\mathbb{E}$-split torus:

$$T' := \left\{ \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_l \end{pmatrix} : A_i \in \mathbb{E}^\times \right\}$$

and the following $\mathbb{F}$-split subtorus of T':

$$S := \left\{ \begin{pmatrix} a_1 I_m & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_l I_m \end{pmatrix} : a_i \in \mathbb{F}^\times \right\},$$

where I_m is the identity matrix in $GL_m(\mathbb{F})$.
After conjugating with an element of G' we may assume that z is in the apartment of T'. Let $t' \in T'$.

Define $E'_{ij} \in G'$ to be the block matrix of $(m \times m)$-matrices with the identity matrix on the i,j-th block and the zero matrix on the other blocks. For $C \in M_m(\mathbb{F})$ we define $(CE')_{ij} \in A$ to be the block matrix of $(m \times m)$-matrices with the matrix C on the i,j-th block and the zero matrix on the other blocks.

Let $\chi' \in X_*(T')$ be the cocharacter in the proof of [MS12, Proposition 2.3], such that $P'_{t'} = P'(\chi')$. For $i \in \{1, \cdots, l\}$ take $n_i \in \mathbb{Z}$ such that for all $e \in \mathbb{E}^\times \subset GL_m(\mathbb{F})$:

$$\chi'(e) = \sum_{i=1}^{l} e^{n_i} E'_{ii}.$$

Define $\chi \in X_*(S)$ by

$$\chi(f) := \sum_{i=1}^{l} f^{n_i} E'_{ii}.$$

CLAIM: $P_{t'} = P(\chi)$.
We postpone the proof of this claim.
Since $S \subset T'$ and z is in the apartment of T', there exists a maximal split torus T in $GL_N(\mathbb{F})$ with z in the apartment of T and $S \subset T$. Because $z \in \mathbb{A}(T)$ and $\chi \in X_*(T)$, we may apply [MS12, Lemma 5.4] to z and $M_{t'}$. Thus

$$U^1(\mathfrak{A}) = (U^1(\mathfrak{A}) \cap U^-_{t'})(U^1(\mathfrak{A}) \cap M_{t'})(U^1(\mathfrak{A}) \cap U^+_{t'}).$$

Proof of CLAIM:
Although the definitions of $M_{t'}, U^{\pm}_{t'}$ are for $t' \in GL_N(\mathbb{F})$, for the proof of the claim it is easier to define their analogues in $M_N(\mathbb{F})$:

$$\mathfrak{p}_g := \{p \in M_N(\mathbb{F}) \mid \{g^n p g^{-n} : n \in \mathbb{N}\} \text{ is bounded}\},$$
$$\mathfrak{m}_g := \{p \in M_N(\mathbb{F}) \mid \{g^z p g^{-z} : z \in \mathbb{Z}\} \text{ is bounded}\}.$$

Note that $\mathfrak{p}_g$ and $\mathfrak{m}_g$ are $\mathbb{F}$-subalgebras and moreover the Lie algebras of P_g and M_g. Define $\mathfrak{p}'_g$ and $\mathfrak{m}'_g$ to be the corresponding sets in G', then $\mathfrak{p}'_g = \mathfrak{p}_g \cap M_l(\mathbb{E})$ and $\mathfrak{m}'_g = \mathfrak{m}_g \cap M_l(\mathbb{E})$.

Let D be the subalgebra of block diagonal matrices with blocks of $(m \times m)$-matrices. We claim $D \subset \mathfrak{m}_{t'}$. To prove this, it is enough to show that matrices in D with only one non-trivial $(m \times m)$-block matrix are in $M_{t'}$. Since $\mathbb{E}^\times$ is compact modulo center in $GL_m(\mathbb{F})$, these matrices lie in $M_{t'}$. Thus $D \subset \mathfrak{m}_{t'}$.

Assume $1 + E'_{ij} \in \mathfrak{p}'_{t'}$. Let $C \in M_m(\mathbb{F})$. Define $d_{C,i} \in D$ to be the block diagonal matrix with the matrix C on the i-th block and the identity matrix on the other blocks on the diagonal. Then $d_{C,i} \in \mathfrak{p}_{t'}$ and $1 + E'_{ij} \in \mathfrak{p}_{t'}$. Thus

$$1 + (CE')_{ij} = d_{C,i}(1 + E'_{ij}) - d_{C,i} \in \mathfrak{p}_{t'}.$$

Thus $1 + (CE')_{ij} \in \mathfrak{p}_{t'}$ for all $C \in M_m(\mathbb{F})$. Hence $P_{t'} = P(\chi)$. $\square$

In order to prove the existence of a cocharacter $\chi \in X_*(S)$ with $P_{t'} = P(\chi)$ one could also take the following 'more sophisticated' shortcut when $\mathbb{F}[\delta]$ is a separable field extension: the characteristic polynomial of $\delta \in GL_m(\mathbb{F})$ is (up to a constant) the same as the minimal polynomial of δ over $\mathbb{F}$. Since $(\mathbb{F}[\delta]/\mathbb{F})$ is separable, the minimal polynomial has different roots, thus the characteristic polynomial of $\delta \in GL_m(\mathbb{F})$ has different roots. Therefore, δ is a regular semisimple element of $GL_m(\mathbb{F})$. Thus $Z_{GL_m(\mathbb{F})}(\delta) = \mathbb{E}^\times$, hence $\mathbb{E}^\times$ is a torus. Hence T' is a torus. By the proof of [MS12, Proposition 2.3], for every $t' \in T'$, there exists a cocharacter defined over $\mathbb{F}$ such that $P_{t'} = P(\chi)$. The image of such a cocharacter must be in the maximal split torus of T', which is S. Hence $\chi \in X_*(S)$.

Lemma 5.9. *Let $\delta \in GL_N(\mathbb{F})$ such that $\mathbb{E} = \mathbb{F}[\delta]$ is a non-trivial field extension of $\mathbb{F}$. Define $G' := Z_G(\delta)$. Let $\mathcal{L}$ be an $\mathcal{O}_{\mathbb{E}}$-lattice chain in $\mathbb{E}^{\frac{N}{[\mathbb{E}:\mathbb{F}]}}$ with $e(\mathcal{L}) = 1$. Then $\mathcal{L}$ is also* *an $\mathcal{O}_{\mathbb{F}}$-lattice chain. Let $z \in \mathcal{B}(GL_N(\mathbb{F}))$ be the corresponding point. Let $s \in \mathbb{R}_{\geq 0}$. Define $L_s := G'_z G_{z,s}$. Then, for $g \in G'$,*

$$|G_z \backslash G_z g G_{[z]} / Z_G L_s| \leq [G_{[z]} : Z_G G_z] q^{\frac{3N^2}{2}} [G_z : L_s]^{\frac{1}{2}}.$$

Proof. The proof of this lemma is based on [KST16, Lemma 4.14].
Let T' be a maximal $\mathbb{E}$-split torus as in Lemma 5.8. Since $e(\mathcal{L}) = 1$, G'_z is a maximal compact subgroup of G'. By the Cartan decomposition, $G' = G'_z T' G'_z$. Since $g \in G'$ and $G'_z \subset G_z$, we may and will assume that $g = t \in T'$. Then we have the following inequalities:

$$\begin{aligned} |G_z \backslash G_z t G_{[z]} / Z_G L_s| &\leq [G_{[z]} : Z_G G_{z,0+}] |G_z \backslash G_z t G_{z,0+} / L_s| \\ &\leq [G_{[z]} : Z_G G_{z,0+}] q^{N^2} [G_z \cap U_t^- : L_s \cap U_t^-] \\ &\leq [G_{[z]} : Z_G G_{z,0+}] q^{N^2} q^{\dim U_t^-} [G_z : L_s]^{\frac{1}{2}} \\ &\leq [G_{[z]} : Z_G G_z] q^{\frac{3N^2}{2}} [G_z : L_s]^{\frac{1}{2}}. \end{aligned}$$

The second and third inequality are consequences of the choice of T' and Lemma 5.8:
The second inequality is due to

$$G_z t G_{z,0+} = G_z t G_{z,0+} t^{-1} t = G_z t (G_{z,0+} \cap M_t U_t^+) t^{-1} t (G_{z,0+} \cap U_t^-) t^{-1} t = G_z t (G_{z,0+} \cap U_t^-)$$

and $[G_z : G_{z,0+}] \leq q^{N^2}$. The third inequality is proved by [KST16, Lemma 3.9]. □

5.5 Intertwining and supercuspidal representations

Let $y \in \mathbb{A}(T^\gamma) \cap \mathcal{B}(G)$.

Let $e_1, \ldots, e_N \in \mathbb{F}^N$ be the standard basis for $\mathbb{F}^N$. Let $e, q \in \mathbb{N}$ with $N = eq$. Define $L(N, e)$ to be the following lattice chain: For $0 \leq r \leq e-1$, define

$$L_r := \left\langle e_1, \ldots, e_{(e-r)q}, \varpi e_{(e-r)q+1}, \ldots, \varpi e_N \right\rangle_{\mathcal{O}}.$$

For $n = ie + r$, with $0 \leq r \leq e-1$, define $L_n := \varpi^i L_r$.

For $e \mid N$, we define $\mathfrak{A}(e)$ to be the order corresponding to the chain of lattices $L(N, e)$. We denote the Jacobson radical of $\mathfrak{A}(e)$ by $\mathfrak{P}_e$.

By conjugation with an element of G, we may assume that the principal order used in the construction of the irreducible supercuspidal representation π is equal to $\mathfrak{A}(e)$.

5.5.1 The V_g reduction

Lemma 5.10 ([KST16, Lemma 4.5])**.** $\theta_\pi(\gamma) = \mathrm{tr}(\pi(\gamma) | V^{G_{y,r+}})$.

Proof. By [Mur03, Lemma 4.2], the level of the representation is equal to $n/e = r$. Now we can apply [KST16, Lemma 4.5]. □

Since $G_{y,r} \subset G_x$ for r large enough, we will consider V as a G_x-representation. By Mackey's formula:

$$\pi|_{G_x} = \bigoplus_{g \in G_x \backslash G / J} \operatorname{Ind}_{G_x \cap {}^g J}^{G_x} {}^g\rho.$$

Therefore, we define the subrepresentation $V_g := \operatorname{Ind}_{G_x \cap {}^g J}^{G_x} {}^g\rho$ for $g \in G$. In view of Lemma 5.10, we want to know the following set:

$$\mathcal{X} := \left\{ g \in G \mid V_g^{G_{y,r+}} \neq 0 \right\}.$$

We will get an upper bound for $|G_x \backslash \mathcal{X} / J|$. In the next two subsections of this section, we show that $\mathcal{X}$ is contained in two double cosets: one related to the group $Z_G(\beta_l)$ and the other related to compact (modulo center).

5.5.2 Intertwining condition

In this subsection we show that whenever $V_g^{G_{y,r+}} \neq 0$, a particular coset of g contains an intertwiner between two strata with the same order. These intertwiners are contained in a certain double coset containing $Z_G(\beta_l)$.

Lemma 5.11. *Let $[\mathfrak{A}, n, n-1, \alpha]$ and $[\mathfrak{A}, n, n-1, \beta]$ be simple strata. Assume that g intertwines these strata. Then*

$$g \in U(\mathfrak{A}) Z_G(\alpha) U(\mathfrak{A}).$$

Proof. Since the strata are intertwined, [BK93, Theorem 2.6.1] says that we can choose $x \in U(\mathfrak{A})$ such that $[\mathfrak{A}, n, n-1, x\alpha x^{-1}]$ is equivalent with $[\mathfrak{A}, n, n-1, \beta]$. Then g intertwines the strata $[\mathfrak{A}, n, n-1, \alpha]$ and $[\mathfrak{A}, n, n-1, x\alpha x^{-1}]$. Thus $x^{-1}g$ intertwines $[\mathfrak{A}, n, n-1, \alpha]$ with itself. According to [BK93, Lemma 1.5.8], then $x^{-1}g \in U^1(\mathfrak{A}) Z_G(\alpha) U^1(\mathfrak{A})$. Thus $g \in U(\mathfrak{A}) Z_G(\alpha) U(\mathfrak{A})$. □

Lemma 5.12. *Let $[\mathfrak{A}(e), n, n-1, \beta]$ be a non-split simple stratum. Let f be the degree of the irreducible polynomial dividing ϕ_β. Then there exist $u \in U(\mathfrak{A}(e))$ such that the representation $\psi_\beta : U^n(\mathfrak{A}) \to \mathbb{C}^\times$ contains the stratum*

$$\left[u\mathfrak{A}(N/f)u^{-1}, \frac{Nn}{fe}, \frac{Nn}{fe} - 1, \beta \right].$$

Proof. By [BK93, Proposition 2.5.11], we can find $x \in U(\mathfrak{A}(e))$ such that $[\mathfrak{A}(e), n, n-1, x\beta x^{-1}]$ is equivalent to a stratum in $\boldsymbol{\gamma}$-standard form [BK93, Definition 2.5.7], say $[\mathfrak{A}(e), n, n-1, \alpha]$. The proof of [BK93, Proposition 2.5.8] shows that, since $[\mathfrak{A}(e), n, n-1, x\beta x^{-1}]$ is a simple stratum, we may assume that $[\mathfrak{A}(e), n, n-1, \alpha]$ is a simple stratum. Then, by the proof of [BK93, Corollary 2.5.9], there exist $y \in U(\mathfrak{A}(e))$ such that $[y\mathfrak{A}(N/f)y^{-1}, \frac{Nn}{fe}, \frac{Nn}{fe} - 1, \alpha]$ is a simple stratum. Since $x\beta x^{-1} - \alpha \in \mathfrak{P}_e^{1-n} \subset y\mathfrak{P}_{N/f}^{1-\frac{Nn}{fe}} y^{-1}$ and $\alpha \in \mathcal{K}(y\mathfrak{A}(N/f)y^{-1})$, also $x\beta x^{-1} \in \mathcal{K}(y\mathfrak{A}(N/f)y^{-1})$. Since $x\beta x^{-1}$ is minimal over $\mathbb{F}$, [BK93, Exercise 1.5.6] shows that $[y\mathfrak{A}(N/f)y^{-1}, \frac{Nn}{fe}, \frac{Nn}{fe} - 1, x\beta x^{-1}]$ is a simple stratum. Thus $[x^{-1}y\mathfrak{A}(N/f)y^{-1}x, \frac{Nn}{fe}, \frac{Nn}{fe} - 1, \beta]$ is a simple stratum contained in ϕ_β, with $x^{-1}y \in U(\mathfrak{A}(e))$. □

We will now give the set-up in which we calculate the intertwining condition.

Let x be the point in the building corresponding to the order $\mathfrak{A}(e)$. Let $[\mathfrak{A}(e), n, n-1, \beta_l]$ be the last stratum in the defining sequence of $[\mathfrak{A}(e), n, 0, \beta]$. Let $d := \gcd(n, e)$.
Fix $u \in U(\mathfrak{A}(e))$ such that $[u\mathfrak{A}(N/f)u^{-1}, \frac{Nn}{fe}, \frac{Nn}{fe} - 1, \beta_l]$ is a simple stratum contained in ϕ_{β_l}.
Let $z \in \mathcal{B}$ be the point corresponding to $\mathfrak{A}(e/d)$. Then $G_x \subset G_z$ and $G_{x,r} \subset G_{z,r}$ with $r = \text{depth}\ \pi$. For $g \in G$, define $W_g := \text{Ind}_{G_z \cap {}^gJ}^{G_z} {}^g\rho$. Then $V_g \subset W_g$.
Let $z' \in \mathcal{B}$ be the point corresponding to $\mathfrak{A}(N/f)$, with f the degree of the irreducible polynomial dividing ϕ_{β_l}.
Let $\mathfrak{A}$ be the order corresponding to the point y.
Without loss of generality, we assume that the order $\mathfrak{A}$ corresponds to a lattice chain $(L_i)_{i\in\mathbb{Z}}$ such that the lattice subchain $(L_{kj})_{j\in\mathbb{Z}}$ corresponds to the order $\mathfrak{A}(e/d)$ of z, where $ke/d = e(\mathfrak{A})$.

Lemma 5.13. *Let ρ be a G_z-subrepresentation of π. If ρ contains the fundamental stratum $[\mathfrak{A}, m, m-1, \alpha]$ with $m/e(\mathfrak{A}) \geq 2$, then it also contains a simple stratum of the form $[u\mathfrak{A}(N/f)u^{-1}, \frac{Nn}{fe}, \frac{Nn}{fe} - 1, \alpha']$, where f is the degree of the irreducible polynomial dividing $\phi_{\alpha'}$.*

Proof. Since the stratum is fundamental of level greater than 1, the representation ρ contains also a fundamental stratum $[\mathfrak{A}(e/d), n/d, n/d - 1, \alpha_1]$. By the proof of [Kut88, Theorem 3.2], there exist α' and $v \in G_z$ such that $[v\mathfrak{A}(N/f')v^{-1}, n', n'-1, v\alpha'v^{-1}]$, with f' the degree of the irreducible polynomial dividing $\phi_{\alpha'}$, is contained in ρ. Since this stratum intertwines with $[u\mathfrak{A}(N/f)u^{-1}, \frac{Nn}{fe}, \frac{Nn}{fe} - 1, \beta_l]$, we have $\phi_{\alpha'} = \phi_{\beta_l}$. Thus, in particular, $f = f'$ and $n' = \frac{Nn}{fe}$. Since $v, u \in G_z$, also $[u\mathfrak{A}(N/f)u^{-1}, n', n'-1, u\alpha'u^{-1}]$ is contained in ρ. □

Theorem 5.14. *If $V_g^{G_{y,r+}} \neq 0$, then $g \in G_z Z_G(\beta_l) G_z$.*

Proof. Since $V_g^{G_{y,r+}} \neq 0$, also $W_g^{G_{y,r+}} \neq 0$. Since $r = \text{depth}(\pi)$, W_g contains a fundamental stratum with order corresponding to y. By Lemma 5.13, the representation W_g contains a simple stratum $[u\mathfrak{A}(N/f)u^{-1}, \frac{Nn}{fe}, \frac{Nn}{fe} - 1, \alpha]$. Thus W_g contains the $G_{uz',r}$-representation ψ_α. By Mackey's formula, W_g is, as $G_{uz',r}$-representation, isomorphic to

$$\bigoplus_{l \in G_{uz',r} \backslash G_z / G_z \cap {}^{lg}J} \text{Ind}_{G_{uz',r} \cap {}^{lg}J}^{G_{uz',r}} {}^{lg}\rho.$$

Without loss of generality, we may assume that ψ_α is a subrepresentation of $\text{Ind}_{G_{uz',r} \cap {}^gJ}^{G_{uz',r}} {}^g\rho$. Thus, by Frobenius reciprocity, ψ_α is, restricted to $G_{uz',r} \cap {}^gJ$, a subrepresentation of ${}^g\rho$. Now by Lemma 5.12, ρ restricted to $G_{uz',r}$ is a sum of copies of ψ_{β_l}. Thus $\psi_\alpha = {}^g\psi_{\beta_l}$ as representations of $G_{uz',r} \cap {}^gG_{uz',r}$. By Lemma 5.11, $g \in G_{uz'} Z_G(\beta_l) G_{uz'}$. Since $G_{uz'} \subset G_z$, the theorem follows. □

5.5.3 The unipotent radical condition

In this subsection, G is any reductive p-adic group.
Regarding the main result of this chapter one could just look at [KST16, Lemma 4.11],

whose proof is also valid for all reductive groups over local non-Archimedean fields. In this subsection we have included another proof of [KST16, Lemma 4.7], which is used to prove [KST16, Lemma 4.11].

Theorem 5.15. *Let J be a open compact modulo center subgroup of G. Let (ρ, V) be a smooth representation of J. Assume that c-$\mathrm{Ind}_J^G \rho$ is an admissible representation of G. Then for every unipotent radical N of a standard parabolic subgroup $P \subsetneq G$,*

$$V^{J \cap N} = 0.$$

Proof. Let $v \in V^{J\cap N}$ be a non-zero vector.
Let S be a maximal split torus contained in P. Take a point x in the apartment of S. Let $r > 0$ be such that $G_{x,r} \subset J$ and $v \in V^{G_{x,r}}$. Take $s \in S$ such that $M := Z_G(s)^0$ is a Levi subgroup of P and $\nu(\alpha(s)) < 0$ for the roots α of S in Lie(N). Take $\overline{N}$ to be the unipotent radical of the parabolic subgroup opposite to P. Then

$$sG_{x,r}s^{-1} = s(G_{x,r} \cap N)s^{-1}s(G_{x,r} \cap M)s^{-1}s(G_{x,r} \cap \overline{N})s^{-1} \subset N(G_{x,r} \cap M)(G_{x,r} \cap \overline{N}) = NG_{x,r}$$

Thus $J \cap sG_{x,r}s^{-1} \subset J \cap (NG_{x,r}) = (J \cap N)G_{x,r}$. Therefore, $v \in V^{J\cap sG_{x,r}s^{-1}}$. We define a vector $f_{v,s}$ in $\left(\text{c-Ind}_J^G \rho\right)^{G_{x,r}}$ as follows:

$$f_{v,s}(g) := \begin{cases} \rho(j)v & \text{for } g = jsk, j \in J, k \in G_{x,r}, \\ 0 & \text{if } g \notin JsG_{x,r}. \end{cases}$$

(This function is well-defined, because $v \in V^{J\cap sG_{x,r}s^{-1}}$.)
Construct a sequence (s_n) in S as follows: Let $s_1 = 1$ and take $s_{n+1} \in S$ such that $s_{n+1}[x] \notin \bigcup_{i=0}^{n} Js_i[x]$, where $[x]$ is the projection of x on the reduced building. Then $Js_iG_{x,r} \cap Js_jG_{x,r} = \emptyset$ if $i \neq j$. Thus the f_{v,s_i} are linearly independent. Therefore, $\left(\text{c-Ind}_J^G \rho\right)^{G_{x,r}}$ is infinite-dimensional. This contradicts the admissibility of c-$\mathrm{Ind}_J^G \rho$. $\square$

Corollary 5.16 ([KST16, Lemma 4.7]). *If c-$\mathrm{Ind}_J^G \rho$ is an irreducible representation of G, then for every unipotent radical N of a standard parabolic subgroup $P \subsetneq G$,*

$$V^{J \cap N} = 0.$$

Proof. Since irreducible smooth representations are admissible, this follows directly. $\square$

The proof in [KST16] uses Frobenius reciprocity and that irreducible representations of J are finite-dimensional.

Let O be the point in the building corresponding to the maximal compact subgroup $GL_N(\mathcal{O})$. Recall that x is the point in the building corresponding to the order $\mathfrak{A}(e)$. Let T be the diagonal torus and C a chamber containing x and O in its closure. Let Δ be the set of positive roots associated with C. Define $r_o = \lceil r + 1 \rceil$.

Theorem 5.17 ([KST16, Lemma 4.11]). *If $V_g^{G_{y,r+}} \neq 0$, then $g \in G_{\mathsf{O}}T(-r_o)G_{\mathsf{O}}$, where*

$$T(-r_o) = \{t \in T \mid \forall[\alpha \in \Delta]\ 1 \leq |\alpha(t^{-1})| \leq q^{r_o+2}\}.$$

Although the setting of [KST16] is for p-adic number fields, their proof is also valid for general reductive groups over a non-Archimedean local field.

5.6 Proof of the KST-conjecture for $GL_N(\mathbb{F}), N \geq 3$

In this section we need Hypothesis 1.

Lemma 5.18. *Let A be a subset of G and H, J, K subgroups of G. Then*

$$|H\backslash HAJ/J| \leq [H : H \cap K] \cdot [K : H \cap K] \cdot |K\backslash KAJ/J|.$$

Proof.

$$\begin{aligned} |H\backslash HAJ/J| &\leq [H : H \cap K] \cdot |(H \cap K)\backslash(H \cap K)AJ/J| \\ &\leq [H : H \cap K] \cdot |(H \cap K)\backslash KAJ/J| \\ &\leq [H : H \cap K] \cdot [K : H \cap K] \cdot |K\backslash KAJ/J|. \end{aligned}$$ □

To simplify the notation of these double cosets, we denote $H\backslash HAJ/J$ by $H\backslash A/J$.

Recall $\pi = \text{c-Ind}_J^G(\rho)$, $V_g := \text{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho$ and

$$\mathcal{X} := \left\{ g \in G \mid V_g^{G_{y,r+}} \neq 0 \right\}.$$

Theorem 5.19. *There exists $C > 0$ depending only on G and μ_{G/Z_G} such that*

$$|G_x\backslash\mathcal{X}/J|\mu_{G/Z_G}(J/Z_G) \leq C(r_o + 3)^N q^{\frac{1}{2}(\dim G' - \dim G)s}.$$

Proof. We may assume that we are in the situation of §5.5.2. Thus the supercuspidal representation π has been constructed with the simple stratum $[\mathfrak{A}(e), n, 0, \beta]$. The point $x \in \mathcal{B}(G)$ is the point corresponding to $\mathfrak{A}(e)$. We defined $s \in \mathbb{R}$ such that $G_{x,s} = U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}(e))$, i.e., $s = \frac{\lfloor \frac{n}{2} \rfloor + 1}{e}$. The last stratum of the defining sequence of $[\mathfrak{A}(e), n, 0, \beta]$ is $[\mathfrak{A}(e), n, r_l, \beta_l]$. Define $d = \gcd(n, e)$, $\mathbb{E} = \mathbb{F}[\beta_l]$ and $G' = Z_G(\beta_l)$.

Let $\mathcal{L} = (L_i)_{i\in\mathbb{Z}}$ be the $\mathcal{O}_\mathbb{E}$-lattice chain corresponding to $\mathfrak{A}(e)$ and hence to x. For $k \mid e(\mathcal{L})$, define the lattice chain $\mathcal{L}_k := (L_{ki})_{i\in\mathbb{Z}}$.

Since β_l is a minimal element of $\mathbb{F}$, $\gcd(\nu_\mathbb{E}(\beta_l), e(\mathbb{E} : \mathbb{F})) = 1$. Since $\mathcal{L}$ is an $\mathcal{O}_\mathbb{E}$-lattice chain,

$$-n = \nu_{\mathfrak{A}(e)}(\beta_l) = \nu_\mathbb{E}(\beta_l)\frac{e}{e(\mathbb{E} : \mathbb{F})}.$$

Therefore,

$$d = \gcd(n, e) = \frac{e}{e(\mathbb{E} : \mathbb{F})} \gcd(\nu_\mathbb{E}(\beta_l), e(\mathbb{E} : \mathbb{F})) = \frac{e}{e(\mathbb{E} : \mathbb{F})}.$$

Thus $\mathcal{L}_d$ has period $e(\mathbb{E} : \mathbb{F})$ over $\mathbb{F}$ and 1 over $\mathbb{E}$. Let z be the point corresponding to the lattice chain $\mathcal{L}_d$. The point O, corresponding to the maximal subgroup $GL_N(\mathcal{O})$, is also the point in the building corresponding to the lattice chain $\mathcal{L}_e$ and order $\mathfrak{A}(0)$. The points x, z and O are contained in the closure of one chamber, say C. Moreover,

$$G_x \subset G_z \subset G_\mathsf{O}.$$

Since these points are all in the closure of one chamber, there exists $D > 0$ (independent of x and $\mathbb{E}$) with

$$\begin{aligned}[G_a : G_a \cap G_b] &\leq D, \\ [G_{[a]} : G_{[a]} \cap G_{[b]}] &\leq D, \\ [G_{[a]} : Z_G G_a] &\leq D, \\ [G'_c : G'_c \cap G'_d] &\leq D,\end{aligned}$$

for all $a, b \in \{x, z, \mathsf{O}\}$ and $c, d \in \{x, z\}$.

By conjugating γ, we may assume that the point $y \in \mathcal{B} \cap \mathbb{A}(T^\gamma)$ lies in the closure of C.

We start our calculations with the following observation:

$$|G_x \backslash \mathcal{X} / J| \leq \sum_{g \in G_z \backslash \mathcal{X} / G_{[z]}} |G_x \backslash G_z g G_{[z]} / J|.$$

Since $|G_{\mathsf{O}} \backslash \mathcal{X} / G_{[\mathsf{O}]}| \leq (r_o + 3)^N$ by [KST16, Lemma 4.11] (Theorem 5.17), then

$$|G_z \backslash \mathcal{X} / G_{[z]}| \leq D^3 (r_o + 3)^N.$$

So to prove Theorem 5.19 it is enough to show that, for all $g \in \mathcal{X}$,

$$|G_x \backslash G_z g G_{[z]} / J| \leq \frac{1}{\mu_{G/Z_G}(J/Z_G)} q^{\frac{5}{2}N^2} D^9 q^{\frac{1}{2}(\dim G' - \dim G)s}.$$

For $X \subset G$, we define $\mu_{G/Z_G}(X) := \mu_{G/Z_G}(X Z_G / Z_G)$. Let $g \in \mathcal{X}$ and let $g' \in G'$ be such that $g \in G_z g' G_z$ (Theorem 5.14). Then we have the following (in)equalities:

$$|G_x \backslash G_z g G_{[z]} / J| = |G_x \backslash G_z g' G_{[z]} / J|$$

and since $J \subset G_{[x]} G_{x,s}$,

$$\begin{aligned}&\leq [G'_{[x]} G_{x,s} : J] \cdot |G_x \backslash G_z g' G_{[z]} / G'_{[x]} G_{x,s}| \\ &\leq [G'_{[x]} G_{x,s} : J] D^4 \cdot |G_x \backslash G_z g' G_{[z]} / G'_{[z]} G_{z,s}| \\ &\leq [G'_{[x]} G_{x,s} : J] D^5 \cdot |G_z \backslash G_z g' G_{[z]} / G'_{[z]} G_{z,s}| \\ &\leq [G'_{[x]} G_{x,s} : J] D^5 \cdot |G_z \backslash G_z g' G_{[z]} / Z_G G'_z G_{z,s}|\end{aligned}$$

applying Lemma 5.9:

$$
\begin{aligned}
&\leq [G'_{[x]}G_{x,s} : J] D^6 q^{\frac{3}{2}N^2} [G_z : G'_z G_{z,s}]^{\frac{1}{2}} \\
&= \frac{\mu_{G/Z_G}(G'_{[x]}G_{x,s})}{\mu_{G/Z_G}(J)} q^{\frac{3}{2}N^2} D^6 [G_z : G'_z G_{z,s}]^{\frac{1}{2}} \\
&\leq \frac{1}{\mu_{G/Z_G}(J)} \mu_{G/Z_G}(G'_{[z]}G_{z,s}) q^{\frac{3}{2}N^2} D^8 [G_z : G'_z G_{z,s}]^{\frac{1}{2}} \\
&\leq \frac{1}{\mu_{G/Z_G}(J)} [G_{[z]} : G'_{[z]}G_{z,s}]^{-1} q^{\frac{3}{2}N^2} D^9 [G_z : G'_z G_{z,s}]^{\frac{1}{2}} \\
&\leq \frac{1}{\mu_{G/Z_G}(J)} [G_z : G'_z G_{z,s}]^{-1} q^{\frac{3}{2}N^2} D^9 [G_z : G'_z G_{z,s}]^{\frac{1}{2}} \\
&= \frac{1}{\mu_{G/Z_G}(J)} q^{\frac{3}{2}N^2} D^9 [G_z : G'_z G_{z,s}]^{-\frac{1}{2}} \\
&\leq \frac{1}{\mu_{G/Z_G}(J)} q^{\frac{3}{2}N^2} D^9 [G_{z,0+} : G'_{z,0+} G_{z,s}]^{-\frac{1}{2}}
\end{aligned}
$$

which is by Lemma 5.5 and $\lfloor s \rfloor - 1 \geq s - 2$

$$
\leq \frac{1}{\mu_{G/Z_G}(J)} q^{\frac{5}{2}N^2} D^9 q^{\frac{1}{2}(\dim G' - \dim G)s},
$$

where most of the inequalities are due to Lemma 5.18 and/or the choice of D. $\square$

Theorem 5.20. *Let $G = GL_N(\mathbb{F})$, with $N > 2$, and $\gamma \in G_{0+}$ a tamely ramified semisimple element of G. For $(\pi_i)_{i=0}^{\infty}$, a sequence of representations satisfying Hypothesis 1,*

$$
\frac{|\theta_{\pi_i}(\gamma)|}{\deg(\pi_i)} \to 0 \text{ as } \deg(\pi_i) \to \infty.
$$

Proof. (The proof is the same as the one given in [KST16, §4.5].)
Let π be a representation satisfying Hypothesis 1, such that $l(\pi) \geq \mathrm{sd}(\gamma)$. We may assume that we have the same setting as in the proof of Theorem 5.19. Since x and y are in the closure of the chamber C, $G_{y,0+} \subset G_x$. Then $\gamma \in G_x$, because $\gamma \in G_{y,0+}$. Let $g \in G$ be such that $V_g^{G_{y,r+}} \neq 0$. Recall $H_g := G_x \cap {}^gJ$.

$$
\mathrm{tr}(\pi(\gamma), \mathrm{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho) = \sum_{k \in H_g \backslash G_x} \widehat{\chi}(k\gamma k^{-1}),
$$

where χ is the character of ${}^g\rho$ and $\widehat{\chi} : G_x \to \mathbb{C}$ the induced character defined by

$$
\widehat{\chi}(k) := \begin{cases} \chi(k) & \text{if } k \in H_g, \\ 0 & \text{otherwise,} \end{cases}
$$

for $k \in G_x$. Thus the trace could only be non-zero if there exists a $k' \in G_x$ such that $k'\gamma k'^{-1} \in H_g$. Since $k' \in G_x$,

$$
\begin{aligned}
\mathrm{tr}(\pi(\gamma), \mathrm{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho) &= \mathrm{tr}(\pi(k\gamma k'^{-1}), \mathrm{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho) \\
&= \sum_{k \in H_g \backslash G_x} \widehat{\chi}(kk\gamma k'^{-1}k^{-1}) \\
&\leq [\psi_{k'\gamma k'^{-1}}^{-1}(H_g) \cap G_x : H_g] \dim \rho.
\end{aligned}
$$

By Corollary 5.7, then:

$$\begin{aligned}\operatorname{tr}(\pi(\gamma), \operatorname{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho) &\leq W|D(\gamma)|^{-1} q^{N(\operatorname{ht}(\Phi)\operatorname{sd}(k'\gamma k'^{-1})+2)} q^{(N-1)s} \dim \rho \\ &= W|D(\gamma)|^{-1} q^{N(\operatorname{ht}(\Phi)\operatorname{sd}(\gamma)+2)} q^{(N-1)s} \dim \rho.\end{aligned}$$

By [KST16, Lemma 4.5] and Theorem 5.19, then

$$\begin{aligned}\left|\frac{\theta_\pi(\gamma)}{\deg(\pi)}\right| &= \frac{\mu_{G/Z_G}(J)}{\dim \rho} \left|\operatorname{tr}(\pi(\gamma)|V_\pi^{G_{y,r+}})\right| \\ &\leq \frac{\mu_{G/Z_G}(J)}{\dim \rho} \sum_{g \in G_x \backslash \mathcal{X}/J} \operatorname{tr}(\pi(\gamma), \operatorname{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho) \\ &\leq \mu_{G/Z_G}(J)|G_x \backslash \mathcal{X}/J| \cdot W|D(\gamma)|^{-1} q^{N(\operatorname{ht}(\Phi)\operatorname{sd}(\gamma)+2)} q^{(N-1)s} \\ &\leq C(r_o+3)^N q^{\frac{1}{2}(\dim G' - \dim G)s} W|D(\gamma)|^{-1} q^{N(\operatorname{ht}(\Phi)\operatorname{sd}(\gamma)+2)} q^{(N-1)s}.\end{aligned}$$

As $\deg(\pi) \to \infty$, on the right-hand side only s and r_o vary, and $s \to \infty$ and $r_o < 2s+2$. Since $\mathbb{F}[\beta_l] \neq \mathbb{F}$, then

$$\dim G - \dim G' = N^2 \left(1 - \frac{1}{[\mathbb{F}[\beta_l] : \mathbb{F}]}\right) \geq \frac{1}{2} N^2.$$

Thus, for some $C' > 0$,

$$\left|\frac{\theta_\pi(\gamma)}{\deg(\pi)}\right| \leq C' s^N (q^{N-1-\frac{1}{4}N^2})^s.$$

If $N \geq 3$, $q^{N-1-\frac{1}{4}N^2} < 1$. Thus $C' s^N (q^{N-1-\frac{1}{4}N^2})^s \to 0$, when $s \to \infty$. □

Theorem 5.21. *Let $G = GL_p(\mathbb{F})$, with p an odd prime, and $\gamma \in G_{0+}$ a tamely ramified semisimple element of G. For all supercuspidal representations π:*

$$\frac{|\theta_\pi(\gamma)|}{\deg(\pi)} \to 0 \text{ as } \deg(\pi) \to \infty.$$

Proof. Since p is prime, a defining sequence for $GL_p(\mathbb{F})$ has only one element. (Otherwise there exists a character of $\mathbb{F}^\times$ lowering the level of π.) Hence in this case Hypothesis 1 holds for all supercuspidal representations. See §5.7 for an overall shorter proof in this case. □

Lemma 5.22. *Let $\mathbb{E}$ be an N-dimensional $\mathbb{F}$-subalgebra of A such that $\mathbb{E}$ is a field. Then the following statements hold:*

1. $\mathcal{L} := \{L \subset V : L$ *is an* $\mathcal{O}_\mathbb{E}$*-lattice*$\}$ *is an* $\mathcal{O}$*-lattice chain.*
2. $\mathfrak{A} := \mathfrak{A}_\mathcal{L}$ *is the unique chain order in* A *with* $\mathbb{E}^\times \subset \mathcal{K}(\mathfrak{A})$*.*
3. *For all* $x \in \mathbb{E}^\times$, $\nu_\mathfrak{A}(x) = \nu_\mathbb{E}(x)$*.*

Proof. See the proof of [BH06, Proposition 12.3]. □

Corollary 5.23. *If $[\mathfrak{A}, n, n-1, \alpha]$ is a simple stratum with $\mathbb{F}[\alpha]$ a field extension of degree N, then* $\gcd(n, e_\mathfrak{A}) = 1$*.*

Proof. Since the stratum is simple, α is minimal over $\mathbb{F}$. Thus $\gcd(\nu_{\mathbb{E}}(\alpha), e(\mathbb{E}:\mathbb{F})) = 1$. Now $\nu_{\mathbb{E}}(\alpha) = \nu_{\mathfrak{A}}(\alpha)$ and $e(\mathbb{E}:\mathbb{F}) = e_{\mathfrak{A}}$. $\square$

Theorem 5.24. *Let $G = GL_{kl}(\mathbb{F})$ with k, l prime and $kl > 8$ and $\gamma \in G_{0+}$ a tamely ramified semisimple element of G. For all supercuspidal representations of G:*

$$\frac{|\theta_\pi(\gamma)|}{\deg(\pi)} \to 0 \text{ as } \deg(\pi) \to \infty.$$

Proof. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum. Since $N := kl$ is a product of 2 primes, the defining sequence of a stratum has at most 2 strata. Let $r_1 := k_0(\beta, \mathfrak{A})$. If $r_1 = n$, then $J \subset G'_{[x]}G_{x,s}$ and the proof of Theorem 5.20 remains valid. So assume that $r_1 < n$. Let $[\mathfrak{A}, n, r_1, \beta_1]$ be the simple stratum equivalent to $[\mathfrak{A}, n, 0, \beta]$. Let $\mathbb{E} := \mathbb{F}[\beta]$. Now

$$J = \mathbb{E}^\times U^{\lfloor \frac{r_1+1}{2} \rfloor}(\mathfrak{B}_{\beta_1}) U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}).$$

Define the subgroup J' of J as follows

$$J' := \mathbb{F}^\times U^{\lfloor \frac{r_1+1}{2} \rfloor}(\mathfrak{B}_{\beta_1}) U^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}).$$

Let $r := \lfloor \frac{n}{2} \rfloor + 1$. For convenience, define the subgroup L_r as

$$L_r := (Z_G(\beta_1) \cap \mathcal{K}(\mathfrak{A}))\, U^r(\mathfrak{A}).$$

Since $\mathbb{E}$ is a maximal field in $M_{kl}(\mathbb{F})$, Lemma 5.22 implies $1 + \mathfrak{p}_{\mathbb{E}}^r \subset U^r(\mathfrak{A})$. Therefore,

$$[J : J'] \leq [\mathbb{E}^\times : \mathbb{F}^\times(1 + \mathfrak{p}_{\mathbb{E}}^r)] \leq e(\mathbb{E}:\mathbb{F})(q_{\mathbb{E}} - 1)q_{\mathbb{E}}^r \leq q_{\mathbb{F}}^N N q_{\mathbb{F}}^{f(\mathbb{E}:\mathbb{F})r}.$$

Now we follow the same line as in the proof of Theorem 5.20. Before we get there we need a similar statement as in Theorem 5.19. Since Theorem 5.19 has been written with the Bruhat-Tits notation of the compact subgroups, we will rewrite the groups J, J' and L_r accordingly. Let x be the point corresponding to the order $\mathfrak{A}$. Let e and e_1 be the periods of the orders $\mathfrak{B}_l$ and $\mathfrak{A}$, respectively. Notice that $e = e(\mathbb{E}:\mathbb{F})$. Let $G' = Z_G(\beta_1)$. Define $s = r/e$ and $s_1 = \lfloor \frac{r_1+1}{2} \rfloor / e_1$, then

$$\begin{aligned} J &= \mathbb{E}^\times G'_{x,s_1} G_{x,s}, \\ J' &= \mathbb{F}^\times G'_{x,s_1} G_{x,s}, \\ L_r &= G'_{[x]} G_{x,s}. \end{aligned}$$

In the language of Bruhat-Tits ($r = es$):

$$[J : J'] \leq q_{\mathbb{F}}^N N q_{\mathbb{F}}^{f(\mathbb{E}:\mathbb{F})e(\mathbb{E}:\mathbb{F})s} = N q_{\mathbb{F}}^{N(s+1)}.$$

We perform the same calculations as for the $GL_N(\mathbb{F})$ case.

$$\begin{aligned}
|G_x \backslash G_z g G_{[z]}/J| &\leq |G_x \backslash G_z g G_{[z]}/J'| \\
&= |G_x \backslash G_z g' G_{[z]}/J'| \\
&\leq [G'_{[x]}G_{x,s} : J'] \cdot |G_x \backslash G_z g' G_{[z]}/G'_{[x]}G_{x,s}| \\
&\leq [G'_{[x]}G_{x,s} : J'] D^4 \cdot |G_x \backslash G_z g' G_{[z]}/G'_{[z]}G_{z,s}| \\
&\leq [G'_{[x]}G_{x,s} : J'] D^5 \cdot |G_z \backslash G_z g' G_{[z]}/G'_{[z]}G_{z,s}| \\
&\leq [G'_{[x]}G_{x,s} : J'] D^5 \cdot |G_z \backslash G_z g' G_{[z]}/Z_G G'_z G_{z,s}| \\
&\leq [G'_{[x]}G_{x,s} : J'] D^6 q^{\frac{3}{2}N^2} [G_z : G'_z G_{z,s}]^{\frac{1}{2}} \\
&= \frac{\mu_{G/Z_G}(G'_{[x]}G_{x,s})}{\mu_{G/Z_G}(J')} q^{\frac{3}{2}N^2} D^6 [G_z : G'_z G_{z,s}]^{\frac{1}{2}} \\
&\leq \frac{1}{\mu_{G/Z_G}(J')} \mu_{G/Z_G}(G'_{[z]}G_{z,s}) q^{\frac{3}{2}N^2} D^8 [G_z : G'_z G_{z,s}]^{\frac{1}{2}} \\
&\leq \frac{1}{\mu_{G/Z_G}(J')} [G_{[z]} : G'_{[z]}G_{z,s}]^{-1} q^{\frac{3}{2}N^2} D^9 [G_z : G'_z G_{z,s}]^{\frac{1}{2}} \\
&\leq \frac{1}{\mu_{G/Z_G}(J')} [G_z : G'_z G_{z,s}]^{-1} q^{\frac{3}{2}N^2} D^9 [G_z : G'_z G_{z,s}]^{\frac{1}{2}} q \\
&= \frac{1}{\mu_{G/Z_G}(J')} q^{\frac{3}{2}N^2} D^9 [G_z : G'_z G_{z,s}]^{-\frac{1}{2}} \\
&\leq \frac{1}{\mu_{G/Z_G}(J')} q^{\frac{5}{2}N^2} D^9 q q^{\frac{1}{2}(\dim G' - \dim G)s} \\
&\leq \frac{1}{\mu_{G/Z_G}(J)} N q^{\frac{5}{2}N^2} D^9 q q^{(\frac{1}{2}(\dim G' - \dim G)+N)s}.
\end{aligned}$$

Therefore,

$$\left| \frac{\theta_\pi(\gamma)}{\deg(\pi)} \right| \leq C(r_o+3)^N q^{(-\frac{1}{4}N^2+N)s} W |D(\gamma)|^{-1} q^{N(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma)+2)} q^{(N-1)s}.$$

Thus it is a constant multiplied by $\left(q^{-\frac{1}{4}N^2+2N-1}\right)^s$. Since $N > 8$, $q^{-\frac{1}{4}N^2+2N-1} < 1$. Thus the right-hand side goes to 0 when s goes to infinity. $\square$

5.7 When N is prime

In this section we prove the KST-conjecture for $GL_N(\mathbb{F})$ when N is prime. By Theorem 5.21 only the case $N = 2$ has not been proven. In this section we give a different proof of Theorem 5.21, which also holds for $N = 2$. To prove the KST-conjecture for a prime we basically follow the same proof as for $N \geq 3$. However, if N is a prime, we can get an other upper bound for $|G_x \backslash \mathcal{X}/J|$ which does not depend on the representation π any more.

Let $y \in \mathcal{B}$ be a point in the apartment of T^γ. Choose x such that y and x are in the closure of a chamber of $\mathcal{B}$.

Since N is prime, there are, up to conjugation, two principal orders in A:

$$\mathfrak{M} := \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & \ddots & \vdots \\ \mathcal{O} & \cdots & \mathcal{O} \end{pmatrix},$$

$$\mathfrak{J} := \begin{pmatrix} \mathcal{O} & \mathfrak{p} & \cdots & \mathfrak{p} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathfrak{p} \\ \mathcal{O} & \cdots & \cdots & \mathcal{O} \end{pmatrix}.$$

Recall that, for $g \in G$, $V_g := \mathrm{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho$ and

$$\mathcal{X} := \{g \in G \mid V_g^{G_{y,r+}} \neq 0\}.$$

Lemma 5.25. *If $V_g^{G_{y,r+}} \neq 0$, then $g \in G_{[x]}$.*

Proof. Assume $x = y$. Let $[\mathfrak{A}, n, n-1, \alpha]$ be the simple stratum of π and the point x.

Thus V_g contains a character of $G_{x,r}$ trivial on $G_{x,r+}$. Let $[\mathfrak{A}, n, n-1, \beta_0]$ be a stratum corresponding to that character. This stratum is fundamental, since $n/e_{\mathfrak{A}} = l(\pi)$. We may now assume that β_0 is of the form b as in the proof of [Kut88, Theorem 3.2], by choosing it appropriately in the coset $\beta_0 + \mathfrak{P}^{1-n}$.
By the proof of [Kut88, Theorem 3.2], there is $h \in G_x$ such that $[\mathfrak{A}, n, n-1, h\beta_0 h^{-1}]$ is a simple stratum. (By Corollary 5.23, we have $R = 1$, hence $f = \overline{f}$.) Let $\beta := h\beta_0 h^{-1}$. Since $h \in G_x$ and $G_{x,r}$ is a normal subgroup of G_x, $[\mathfrak{A}, n, n-1, \beta]$ is a simple stratum contained in V_g. By Mackey's formula

$$\mathrm{Res}_{G_{x,r}}^{G_x} V_g \cong \bigoplus_{h \in G_{x,r} \backslash G_x / G_x \cap {}^gJ} \mathrm{Ind}_{G_{x,r} \cap {}^{hg}J}^{G_{x,r}} {}^{hg}\rho.$$

Now ψ_β is a $G_{x,r}$-subrepresentation of V_g, so without loss of generality we may assume that ψ_β is a subrepresentation of

$$\mathrm{Ind}_{G_{x,r} \cap {}^gJ}^{G_{x,r}} {}^g\rho.$$

The representation ρ, restricted to $G_{x,r}$, is a direct sum of ψ_α. Thus ${}^g\rho$ restricted to $G_{gx,r}$ is a direct sum of ${}^g\psi_\alpha$. By Frobenius reciprocity, ψ_β restricted to $G_{x,r} \cap G_{gx,r}$ is a subrepresentation of such a direct sum, hence $\psi_\alpha = {}^g\psi_\beta$ on $G_{x,r} \cap G_{gx,r}$. So g intertwines the simple strata $[\mathfrak{A}, n, n-1, \alpha]$ and $[\mathfrak{A}, n, n-1, \beta]$. By Lemma 5.11 and $Z_G(\alpha) = \mathbb{E}^\times \subset \mathcal{K}(\mathfrak{A}) = G_{[x]}$, then $g \in G_{[x]}$.

Assume $x \neq y$. If x corresponds to $\mathfrak{J}$, then, by Corollary 5.23, $l(\pi) \in \frac{1}{N}\mathbb{N}$ and $l(\pi) \notin \mathbb{N}$. Since N is prime, the strata with level $l(\pi)$ have an order conjugate to $\mathfrak{J}$. Thus we are left with the case that x corresponds to the stratum $[\mathfrak{M}, n, n-1, \alpha]$ and y corresponds to the order $\mathfrak{A}$ with period $e = e_{\mathfrak{A}}$.

Let $[\mathfrak{A}, en, en-1, \beta]$ be the stratum appearing in $\mathrm{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho$. Take δ such that the stratum $[\mathfrak{M}, n, n-1, \delta]$ appears in $\mathrm{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho$, with $\psi_\beta|_{U_{\mathfrak{A}}^{en}} = \psi_\delta$. (The group $U_{\mathfrak{M}}^n / U_{\mathfrak{A}}^{en+1}$ is abelian and $U_{\mathfrak{M}}^n \subset G_y$.) Since $G_{x,r+} = U_{\mathfrak{M}}^{n+1} \subset U_{\mathfrak{A}}^{en+1} = G_{y,r+}$:

$$\left(\mathrm{Ind}_{G_x \cap {}^gJ}^{G_x} {}^g\rho\right)^{G_{x,r+}} \neq \emptyset.$$

Thus we are in the first case. Hence $g \in G_{[x]}$. $\square$

Corollary 5.26. *There exists a $C > 0$ independent of π such that $|G_x \backslash \mathcal{X} / J| \leq C$.*

Proof. Since $\mathcal{X} = G_{[x]}$ and $Z_G \subset J$, we could take C to be the maximum of $[\mathcal{K}(U(\mathfrak{M})) : Z_G U(\mathfrak{M})]$ and $[\mathcal{K}(U(\mathfrak{J})) : Z_G U(\mathfrak{J})]$. $\square$

Theorem 5.27. *For $G = GL_N(\mathbb{F})$, with N a prime, and $\gamma \in G_{0+}$ a tamely ramified semisimple element of G:*

$$\frac{|\theta_\pi(\gamma)|}{\deg(\pi)} \to 0 \text{ as } \deg(\pi) \to \infty.$$

Proof. As in the proof of Theorem 5.20, by [KST16, Lemma 4.5], Corollary 5.7 and Corollary 5.26,

$$\begin{aligned}
\left|\frac{\theta_\pi(\gamma)}{\deg(\pi)}\right| &= \frac{\mu_{G/Z_G}(J)}{\dim \rho} \left|\mathrm{tr}(\pi(\gamma)|V_\pi^{G_{y,r+}})\right| \\
&\leq \mu_{G/Z_G}(J) |G_x \backslash \mathcal{X} / J| \cdot W |D(\gamma)|^{-1} q^{N(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma)+2)} q^{(N-1)s} \\
&\leq C q^{(\dim G' - \dim G)s} W |D(\gamma)|^{-1} q^{N(\mathrm{ht}(\Phi)\mathrm{sd}(\gamma)+2)} q^{(N-1)s},
\end{aligned}$$

where the last inequality is due to $\mu_{G/Z_G}(J) \leq \frac{D}{[G_x : G'_{x,s} G_{x,s}]}$ for some constant $D > 0$, independent of J, and Lemma 5.5.
On the right-hand side only s varies as $\deg(\pi) \to \infty$. Since $\mathbb{F}[\beta] \neq \mathbb{F}$, then

$$\dim G - \dim G' = N^2 \left(1 - \frac{1}{[\mathbb{F}[\beta] : \mathbb{F}]}\right) \geq \frac{1}{2} N^2.$$

Thus for some $C' > 0$,

$$\left|\frac{\theta_\pi(\gamma)}{\deg(\pi)}\right| \leq C' s^N (q^{N - \frac{1}{2}N^2 - 1})^s.$$

Since $N \geq 2$, $q^{N - \frac{1}{2}N^2 - 1} < 1$. Thus $C' s^N (q^{N - \frac{1}{2}N^2 - 1})^s \to 0$, when $s \to \infty$. $\square$

Chapter 6

Distance to Fixed Points

In this chapter, $\mathcal{B}$ is the reduced building of a reductive p-adic group G.
In this chapter we will give a proof of Theorem 6.1.

Theorem 6.1. *There exists a constant C, depending only on the root datum of G, such that for all compact $g \in G$, $p \in \mathcal{B}$ there exists a $p_0 \in \mathcal{B}^g$ such that*

$$d(p, p_0) \leq Cd(p, gp).$$

Theorem 6.1 was conjectured by Cheng-Chiang Tsai on the forum mathoverflow: http://mathoverflow.net/questions/209527. Tsai formulated this Theorem with the following application in mind:

Corollary 6.2. *Suppose that $A > 0$ is such that for all semisimple regular compact $\gamma \in G$ and $x \in \mathcal{B}$:*

$$|D(\gamma)|^{\frac{1}{2}} \int_{G/T^\gamma} 1_{G_x}(g\gamma g^{-1}) \mathrm{d}g \leq A.$$

Let $C' > 0$ be such that the number of simplices which contain a point with distance at most r from x is bounded by $q^{C'r}$. Then, for all semisimple regular compact $\gamma \in G$ and $h \in G$:

$$|D(\gamma)|^{\frac{1}{2}} \int_{G/T^\gamma} 1_{G_x h G_x}(g\gamma g^{-1}) \mathrm{d}g \leq A q^{C'Cd(x,hx)}.$$

Proof. Assume $\gamma = k_1 h k_2$ with $k_i \in G_x$. Then $d(x, \gamma x) = d(x, k_1 hx) = d(x, hx)$. Therefore, by Theorem 6.1, there exists $y \in \mathcal{B}$ such that $d(x, y) \leq Cd(x, hx)$ and $\gamma \in G_y$. Let Σ be the collection of simplices of $\mathcal{B}$ which contain a point with distance from x at most r. Let B_v be the set of centers of the simplices of Σ. Then $1_{G_x h G_x}(g\gamma g^{-1}) \leq \sum_{y \in B_v} 1_{G_y}(g\gamma g^{-1})$ for all $g \in G$. The conjecture follows. $\square$

Corollary 6.2 enables us to extend known estimates of orbital integrals for 1_{G_x} to indicator functions of G_x-double cosets.

Corollary 6.3. *If the characteristic of the residue class field of $\mathbb{F}$ is large enough, then*

$$|D(\gamma)|^{\frac{1}{2}} \int_{G/T^\gamma} 1_{G_x h G_x}(g\gamma g^{-1}) \mathrm{d}g \leq q^{C_0 (\dim G)^2 \mathrm{rank}(G)} q^{C'Cd(x,hx)},$$

where C_0 is the C in [Tsa15, Corollary 5.6].

Proof. Follows directly from Corollary 6.2 and [Tsa15, Corollary 5.6]. □

R. Cluckers, J. Gordon and I. Halupczok established a similar estimate, see [ST16, Theorem 14.2], by different methods. The constants in Corollary 6.3 are more explicit, but [ST16, Theorem 14.2] holds for all semisimple elements $\gamma \in G$.

First we will prove Theorem 6.1 in the case that G is $\mathbb{F}$-split and almost simple. Than we will show that it holds for all $\mathbb{F}$-split groups. Finally, we complete the proof of Theorem 6.1 by showing that, for every field extension $\mathbb{E}$ of $\mathbb{F}$, the g-fixed point closest to p in $\mathcal{B}(G,\mathbb{E})$ is in $\mathcal{B}(G,\mathbb{F})$.

6.1 The almost simple $\mathbb{F}$-split case

In this section, we assume that G is $\mathbb{F}$-split and almost simple.

Lemma 6.4. *For every $r \in \mathbb{R}_{>0}$ there exists a colored simplicial complex Δ realized in $\mathcal{B}$ such that*

1. *Δ is a building.*
2. *$\mathcal{B} = \bigcup_{\sigma\in\Delta} \sigma$.*
3. *The action of G preserves the colors of Δ.*
4. *The diameter of a chamber in Δ is smaller than r.*

Proof. Let $\mathbb{E}$ be a finite field extension of $\mathbb{F}$. Let $\mathcal{B}(\mathbb{E})$ be the building of $G(\mathbb{E})$ and $\Delta(\mathbb{E})$ its corresponding simplicial complex. Now define $\Delta_{\mathbb{F}}^{\mathbb{E}}$ to be the subcomplex of $\Delta(\mathbb{E})$ consisting of simplices contained in $\mathcal{B}$. For a simplex A of $\Delta_{\mathbb{F}}^{\mathbb{E}}$, define A' to be the simplex of $\mathcal{B}$ containing A of lowest dimension. Now $\Delta_{\mathbb{F}}^{\mathbb{E}}$ is a building (definition [Bro89]):
(B0) An apartment of $\Delta_{\mathbb{F}}^{\mathbb{E}}$ is an apartment of $\Delta(\mathbb{E})$. Thus each apartment of $\Delta_{\mathbb{F}}^{\mathbb{E}}$ is a Coxeter complex.
(B1) Let A and B be simplices of $\Delta_{\mathbb{F}}^{\mathbb{E}}$. The apartment of $\mathcal{B}$ containing A' and B' is also an apartment of $\Delta_{\mathbb{F}}^{\mathbb{E}}$. Therefore, for each pair of simplices of $\Delta_{\mathbb{F}}^{\mathbb{E}}$ there exists an apartment containing both of them.
(B2) Let Σ and Σ' be two apartments containing the simplices A and B. Since $\mathcal{B}(\mathbb{E})$ is a building, there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.
The simplicial complex $\Delta_{\mathbb{F}}^{\mathbb{E}}$ is colored because buildings are colored.

First we give a finite field extension $\mathbb{E}_0$ of $\mathbb{F}$ such that the coloring of $\Delta_{\mathbb{F}}^{\mathbb{E}_0}$ is preserved by G.
Let $\pi : \tilde{G} \to G$ be the algebraic simply connected cover of G. Let T be a maximal $\mathbb{F}$-split torus. Let $\mathbb{A}$ be its apartment and $\overline{W}$ the image of $N_G(T)$ in the set of affine actions on $\mathbb{A}$. Let W be the affine Weyl group generated by the simple reflections of $\mathbb{A}$. Then W of $\mathbb{A}$ is a normal subgroup of $\overline{W}$ with finite index. The affine Weyl group W preserves the colors of $\mathbb{A}$. Let $n_1, \dots, n_m$ be representatives of $\overline{W}/W$ in $N_G(T)$. The map $\pi : \tilde{G}(k) \to G(k)$ is surjective, where k is an algebraic closure of $\mathbb{F}$. Let $\mathbb{E}_0$ be a finite field extension of $\mathbb{F}$ such that $n_i \in \pi(\tilde{G}(\mathbb{E}_0))$. Since $\tilde{G}$ is simply connected, $W_{\tilde{G}} = \overline{W}_{\tilde{G}}$ by [Tit79, 1.13]. Thus

$\tilde{G}$ preserves the colors of $\mathcal{B}(\tilde{G}, \mathbb{E}_0)$. Since π is a central isogeny, $\mathcal{B}(\tilde{G}, \mathbb{E}_0) \cong \mathcal{B}(G, \mathbb{E}_0)$ as metric $\tilde{G}$-spaces ([BT84, 4.2.18]). Therefore, $\pi(\tilde{G}(\mathbb{E}_0))$ preserves the colors of $\mathcal{B}(G, \mathbb{E}_0)$. Hence the n_i's are color-preserving on $\mathcal{B}(G, \mathbb{E}_0)$. Therefore, $\overline{W}$ preserves the colors of $\mathbb{A}(\mathbb{E}_0)$. Thus G preserves the coloring on $\Delta_{\mathbb{F}}^{\mathbb{E}_0}$. In fact, G preserves the colors on $\Delta_{\mathbb{F}}^{\mathbb{E}}$ for every finite field extension $\mathbb{E}$ of $\mathbb{E}_0$.

Let R be the diameter of a chamber in $\mathcal{B}$. Let $\mathbb{E}$ be a field extension of $\mathbb{F}$ containing $\mathbb{E}_0$ with ramification index greater than rR. Then the diameter of a chamber in $\Delta_{\mathbb{F}}^{\mathbb{E}}$ is $\frac{R}{rR} = \frac{1}{r}$.

Thus $\Delta_{\mathbb{F}}^{\mathbb{E}}$ is an example of a simplicial complex satisfying the four properties. $\square$

Proposition 6.5. *Let G be a $\mathbb{F}$-split almost simple group. There exists a constant C such that for all compact $g \in G$ and $p \in \mathcal{B}$, there exists a $p_0 \in \mathcal{B}^g$ such that*

$$d(p, p_0) \leq C d(p, gp).$$

Proof. Let p_0' be the g-fixed point closest to p. Such a point exists since $\mathcal{B}^g$ is closed and convex. If p is fixed by g we are done. Thus, without loss of generality, we assume that $d(p_0', p) > 0$.
Let Δ be the simplicial complex realized in $\mathcal{B}$ fulfilling the four properties of Lemma 6.4 for $r = d(p_0', p)/2$. In this proof we will regard $\mathcal{B}$ as a building with simplicial structure Δ. Thus the chambers and vertices of $\mathcal{B}$ are chambers and vertices of Δ.

Assume that p is in the interior of a chamber, say C_p.

Let p_0 be a vertex fixed by g and C a chamber with p_0 on the boundary. Assume that $d(C, C_p)$ is minimal among such pairs. ($d(C, C_p)$ is the length of the minimal gallery between C and C_p.) Then $d(gC, gC_p)$ is also minimal among such pairs. Let $\mathbb{A}$ be an apartment that contains C and gC.

Lemma 6.6. $gC \cap C = \{p_0\}$.

Proof. Assume that $gC \cap C$ contains a vertex $p' \neq p_0$. Since G is color-preserving, g must fix $gC \cap C$ pointwise. Thus $gp' = p'$. Let C' be the next chamber in the minimal gallery from C to C_p. Since $d(p, p_0)$ is greater than the diameter of a chamber, $C \neq C_p$. If $p_0 \in C'$, then $d(C', C_p) = d(C, C_p) - 1$ for (p_0, C'). This is in contradiction with the minimality of $d(C, C_p)$. Thus $p_0 \notin C'$. Then p' is a vertex of C' because the vertices of C are those of $C \cap C'$ and p_0 (the root system of the Weyl group of G is irreducible). However, then the pair (p', C') leads to a contradiction with the minimality of $d(C, C_p)$. Thus $gC \cap C$ contains no other vertices than p_0. Hence $gC \cap C = \{p_0\}$. $\square$

Let ρ_C (resp. ρ_{gC}) be the retraction to $\mathbb{A}$ centered at C (resp. gC).

Lemma 6.7. $\rho_{gC}(gC_p) = \rho_C(gC_p)$.

Proof. Let $\mathrm{ch}(\mathbb{A}, p_0)$ be the set of chambers in $\mathbb{A}$ containing p_0. Define, for $D \in \mathrm{ch}(\mathbb{A}, p_0)$, the Weyl chamber corresponding to (D, p_0) to be the set of chambers in $\mathbb{A}$ closer to D than to any other chamber in $\mathrm{ch}(\mathbb{A}, p_0)$. The Weyl chamber is a fundamental domain for the action of the stabilizer in the Coxeter group of $\mathbb{A}$ of p_0. Now $\rho_{gC}(gC_p)$ is in the Weyl chamber W_{gC} corresponding with gC and p_0 because $d(gC, gC_p)$ is minimal among the chambers in $\mathrm{ch}(\mathbb{A}, p_0)$. Let Γ be a minimal gallery from C to gC and Γ' be a minimal gallery from gC to $\rho_{gC}(gC_p)$. Every wall crossed by Γ does not go through the Weyl chamber W_{gC}. Since $\rho_{gC}(gC_p)$ is in the Weyl chamber corresponding with gC and p_0, Γ joint with Γ' is a minimal gallery from C to $\rho_{gC}(gC_p)$. Therefore, there exists a minimal gallery from C to gC_p passing through gC. So $\rho_{gC}(gC_p) = \rho_C(gC_p)$. $\square$

Examples. In Figure 6.1 there are two examples in a building of a group of type B_2. In both examples we see a part of an apartment $\mathbb{A}$ containing C and gC. The red lines are the boundaries of the Weyl chamber corresponding with C and p_0 and the blue lines are the boundaries of the Weyl chamber corresponding with gC and p_0. In yellow the images of the chambers C_p and gC_p under the retractions ρ_C and ρ_{gC}.

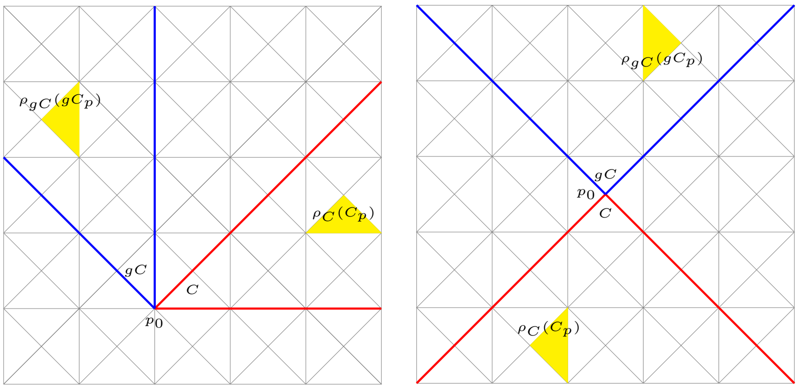


Figure 6.1: Two examples in the apartment $\mathbb{A}$

A corollary of the proof is that there exists an apartment which contains C, gC and gC_p. However, sometimes there is no apartment containing the four chambers C, gC, C_p, gC_p. To demonstrate this we construct an example. Let T be a maximal $\mathbb{F}$-split torus and $\mathbb{A}$ its apartment in $\mathcal{B}$. Let $w \in N_G(T)$ be such that it acts on $\mathbb{A}$ by a rotation of 90 degrees around the point p_0. Let $u \in U^+$ be such that its set of fixed points in $\mathbb{A}$ is the half apartment containing p_0 and bounded by the horizontal wall just above p_0. Let $g = uw$. Now p_0 is the only g-fixed point in the building. Let $C_p = \rho_C(C_p)$ be the chamber such as in the left picture of Figure 6.1. Then gC_p,C_p and gC are not in one apartment.

Continuation of the proof of Proposition 6.5. Since $\rho_C(gC_p) = \rho_{gC}(gC_p)$, in particular $\rho_C(gp) = \rho_{gC}(gp)$. Since $C \cap gC = \{p_0\}$, also the intersection of their corresponding Weyl chambers W_C and W_{gC} only consists of $\{p_0\}$. For l_1, l_2 half lines starting at p_0 going to W_C resp. W_{gC}, define $\theta(l_1, l_2)$ to be the angle between them. Since $W_C \cap W_{gC} = \{p_0\}$, there exists $\Theta > 0$ such that $\pi \geq \theta(l_1, l_2) \geq \Theta$ for all such lines l_1, l_2. Now take l_1 to be the line from p_0 to $\rho_C(p)$ and l_2 the line from p_0 to $\rho_C(gp)$. Since

$$d(p_0, \rho_C(p)) = d(p_0, p) = d(p_0, gp) = d(p_0, \rho_C(gp))$$

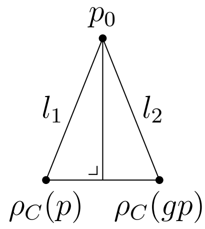


Figure 6.2: The isosceles triangle $p_0, \rho_C(p), \rho_C(gp)$ with the line which is both the angle bisection and the median through p_0

and $0 \leq \frac{\theta}{2} \leq \frac{\pi}{2}$, we get (see Figure 6.2)

$$d(p_0, p) = \frac{d(\rho_C(p), \rho_C(gp))}{2 \sin \frac{\theta(l_1, l_2)}{2}} \leq \frac{1}{2 \sin \frac{\Theta}{2}} d(p, gp).$$

This proves Proposition 6.5 if p lies in the interior of a chamber.

If p is not in the interior of a chamber, then we take a sequence of points p_n in an apartment such that $d(p_n, \mathcal{B}^g) > d(p_0', p)/2$ and $\lim_{n \to \infty} d(p, p_n) = 0$ and p_n is in the interior of a chamber. Let p_n^o the point in $\mathcal{B}^g$ closest to p_n. Then $(p_n^o)_n$ converges to the closest point p^o of $\mathcal{B}^g$ to p. Therefore, we get the same inequality for p. □

Corollary 6.8. *If G is an $\mathbb{F}$-split reductive group, then Theorem* 6.1 *holds.*

Proof. Let G^{ad} be the adjoint group of G and let $\mathrm{Ad} : G \to G^{\mathrm{ad}}$ be the adjoint map. Let $\mathrm{Ab} : G \to G_{\mathrm{ab}}$ with $G_{\mathrm{ab}} = G/[G, G]$ a torus. Now $\mathrm{Ad} \times \mathrm{Ab} : G \to G^{\mathrm{ad}} \times G_{\mathrm{ab}}$ is a central isogeny. Since G^{ad} is an adjoint group it is isomorphic to $\prod_{i=1}^n G_i$, where G_i are simple adjoint groups.

By [BT84, 4.2.18], the extended Bruhat-Tits buildings $\mathcal{B}^e(G)$ and $\mathcal{B}^e(G_{\mathrm{ab}}) \times \prod_{i=1}^n \mathcal{B}^e(G_i)$ are isomorphic as metric G-spaces. Therefore, the reduced buildings are also isomorphic as metric G-spaces. Since Theorem 6.1 holds for the G_i it also holds for $G^{\mathrm{ad}} \times G_{\mathrm{ab}}$. Since the action and distances are preserved by an isomorphism between the buildings $\mathcal{B}^e(G)$ and $\mathcal{B}^e(G_{\mathrm{ab}}) \times \prod_{i=1}^n \mathcal{B}^e(G_i)$ compatible with the isogeny $\mathrm{Ad} \times \mathrm{Ab}$, Theorem 6.1 also holds for G. □

6.2 The general case

Lemma 6.9. *Let $\mathbb{E} : \mathbb{F}$ be a finite field extension. Let $g \in G(\mathbb{F})$ be compact, $p \in \mathcal{B}(\mathbb{F})$ and let p_0 be the point in $\mathcal{B}(\mathbb{E})^g$ closest to p. Then $p_0 \in \mathcal{B}(\mathbb{F})$.*

Proof. Let $\pi(p_0)$ be the point in $\mathcal{B}(\mathbb{F})$ closest to p_0.
Then $\pi(p_0)$ is also fixed by g, thus

$$d(p, p_0) \leq d(p, \pi(p_0)).$$

Assume that $p_0 \neq \pi(p_0)$. Then $p \neq p_0, \pi(p_0)$, otherwise $p = p_0 = \pi(p_0)$.
Now we use the CAT(0)-property of the building $\mathcal{B}(\mathbb{E})$:
Let $\overline{p}, \overline{p_0}, \overline{\pi(p_0)} \in \mathbb{R}^2$, such that $x \mapsto \overline{x}$ for $x = p, p_o, \pi(p_0)$ is distance preserving.
Since $d(p_0, p) \leq d(\pi(p_0), p)$, there is $z \in (\overline{\pi(p_0)}, \overline{p}]$ such that $d_{\mathbb{R}^2}(z, \overline{p_0}) < d_{\mathbb{R}^2}(\overline{\pi(p_0)}, \overline{p_0})$ (see Figure 6.3).

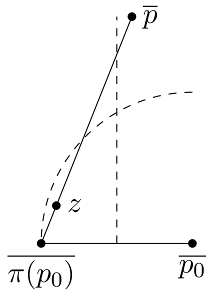


Figure 6.3: The point z

Let $m' \in [\pi(p_0), p]$ be the point in the building corresponding to z. Then

$$d(m', p_0) \leq d_{\mathbb{R}^2}(z, \overline{p_0}) < d(\pi(p_0), p_0),$$

because the building is a CAT(0)-space. Since $m' \in \mathcal{B}(\mathbb{F})$, this is a contradiction with $\pi(p_0)$ being the point in $\mathcal{B}(\mathbb{F})$ closest to p_0. Thus $p_0 = \pi(p_0) \in \mathcal{B}(\mathbb{F})$. □

Proof of Theorem 6.1. Let $\mathbb{E}$ be a finite field extension of $\mathbb{F}$ such that G is $\mathbb{E}$-split. Let C be a constant promised by Corollary 6.8 for $G(\mathbb{E})$. Let $g \in G$ be compact and $p \in \mathcal{B}(\mathbb{F})$. Let p_0 be the point in $\mathcal{B}(\mathbb{E})^g$ closest to p. Then

$$d(p, p_0) \leq C d(p, gp).$$

By Lemma 6.9 $p_0 \in \mathcal{B}(\mathbb{F})$. Thus the C for $G(\mathbb{E})$ also works for G. □

Below a table of the constant C for some one- and two-dimensional buildings:

type	Θ	$C = \frac{1}{2\sin(\frac{\Theta}{2})}$
A_1	π	$\frac{1}{2}$
A_2	$\frac{\pi}{3}$	1
B_2	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2-\sqrt{2}}}$
G_2	$\frac{\pi}{6}$	$\frac{2}{\sqrt{6}-\sqrt{2}}$

The following estimate for simple Coxeter groups has been conjectured by G. Heckman in a conversation with the author: $C = \sin(\frac{\pi}{2h})$, where h is the height of the Coxeter group.

Bibliography

[Adl98] Jeffrey D. Adler. Refined anisotropic K-types and supercuspidal representations. *Pacific J. Math.*, 185(1):1–32, 1998.

[BH96] Colin J. Bushnell and Guy Henniart. Local tame lifting for $\mathrm{GL}(N)$. I. Simple characters. *Inst. Hautes Études Sci. Publ. Math.*, (83):105–233, 1996.

[BH06] Colin J. Bushnell and Guy Henniart. *The local Langlands conjecture for* $\mathrm{GL}(2)$, volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[BK93] Colin J. Bushnell and Philip C. Kutzko. *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, volume 129 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993.

[BL02] Paul Broussous and Bertrand Lemaire. Building of $\mathrm{GL}(m, D)$ and centralizers. *Transform. Groups*, 7(1):15–50, 2002.

[Bro89] Kenneth S. Brown. *Buildings*. Springer-Verlag, New York, 1989.

[BT72] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local. *Inst. Hautes Études Sci. Publ. Math.*, (41):5–251, 1972.

[BT84] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Inst. Hautes Études Sci. Publ. Math.*, (60):197–376, 1984.

[Car79] Pierre Cartier. Representations of p-adic groups: a survey. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, RI, 1979.

[Car85] Roger W. Carter. *Finite groups of Lie type*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1985. Conjugacy classes and complex characters, A Wiley-Interscience Publication.

[Cas77] William A. Casselman. Characters and Jacquet modules. *Math. Ann.*, 230(2):101–105, 1977.

[CGH14] Raf Cluckers, Julia Gordon, and Immanuel Halupczok. Local integrability results in harmonic analysis on reductive groups in large positive characteristic. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(6):1163–1195, 2014.

[Con09] William Conley. Types for supercuspidal representations of $GL(N)$. *http://www.math.ucla.edu/∼wconley/research/simpletypes.pdf*, 2009.

[DeB02a] Stephen DeBacker. Homogeneity results for invariant distributions of a reductive p-adic group. *Ann. Sci. École Norm. Sup. (4)*, 35(3):391–422, 2002.

[DeB02b] Stephen DeBacker. Some applications of Bruhat-Tits theory to harmonic analysis on a reductive p-adic group. *Michigan Math. J.*, 50(2):241–261, 2002.

[Dij72] Gerrit van Dijk. Computation of certain induced characters of p-adic groups. *Mathematische Annalen*, 199(4):229–240, 1972.

[DK00] Johannes J. Duistermaat and Johan A. C. Kolk. *Lie groups*. Universitext. Springer-Verlag, Berlin, 2000.

[Gou97] Fernando Q. Gouvêa. *p-adic numbers*. Universitext. Springer-Verlag, Berlin, second edition, 1997. An introduction.

[HC70] Harish-Chandra. *Harmonic analysis on reductive p-adic groups*. Lecture notes in mathematics. Springer, Berlin, 1970. Notes by G. van Dijk.

[HC99] Harish-Chandra. *Admissible invariant distributions on reductive p-adic groups*, volume 16 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999. Preface and notes by Stephen DeBacker and Paul J. Sally Jr.

[How74] Roger Howe. The Fourier transform and germs of characters (case of Gl_n over a p-adic field). *Math. Ann.*, 208:305–322, 1974.

[HS85] Derek F. Holt and Nicolas Spaltenstein. Nilpotent orbits of exceptional Lie algebras over algebraically closed fields of bad characteristic. *J. Austral. Math. Soc. Ser. A*, 38(3):330–350, 1985.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[Jan04] Jens Carsten Jantzen. Nilpotent orbits in representation theory. In *Lie theory*, volume 228 of *Progr. Math.*, pages 1–211. Birkhäuser Boston, Boston, MA, 2004.

[KST16] Ju-Lee Kim, Sug W. Shin, and Nicolas Templier. Asymptotics and local constancy of characters of p-adic groups. *arXiv:1602.01758v1*, 2016.

[Kut88] Philip C. Kutzko. Towards a classification of the supercuspidal representations of GL_N. *J. London Math. Soc. (2)*, 37(2):265–274, 1988.

[Lem96] Bertrand Lemaire. Intégrabilité locale des caractères-distributions de $\mathrm{GL}_N(F)$ où F est un corps local non-archimédien de caractéristique quelconque. *Compositio Math.*, 100(1):41–75, 1996.

[Lem05] Bertrand Lemaire. Intégrabilité locale des caractères de $\mathrm{SL}_n(D)$. *Pacific J. Math.*, 222(1):69–131, 2005.

[McN04] George J. McNinch. Nilpotent orbits over ground fields of good characteristic. *Math. Ann.*, 329(1):49–85, 2004.

[Moy00] Allen Moy. Displacement functions on the Bruhat-Tits building. In *The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998)*, volume 68 of *Proc. Sympos. Pure Math.*, pages 483–499. Amer. Math. Soc., Providence, RI, 2000.

[MP94] Allen Moy and Gopal Prasad. Unrefined minimal K-types for p-adic groups. *Invent. Math.*, 116(1-3):393–408, 1994.

[MS10] Ralf Meyer and Maarten Solleveld. Resolutions for representations of reductive p-adic groups via their buildings. *J. Reine Angew. Math.*, 647:115–150, 2010.

[MS12] Ralf Meyer and Maarten Solleveld. Characters and growth of admissible representations of reductive p-adic groups. *J. Inst. Math. Jussieu*, 11(2):289–331, 2012.

[Mur03] Fiona Murnaghan. Germs of characters of admissible representations of p-adic general linear groups. *J. Inst. Math. Jussieu*, 2(3):409–481, 2003.

[PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[Rao72] Ramaswamy Ranga Rao. Orbital integrals in reductive groups. *Annals of Mathematics*, 96(3):505–510, 1972.

[Ren10] David Renard. *Représentations des groupes réductifs p-adiques*, volume 17 of *Cours Spécialisés*. Société Mathématique de France, Paris, 2010.

[RJ67] Roger W. Richardson Jr. Conjugacy Classes in Lie Algebras and Algebraic Groups. *Annals of Mathematics*, 86(1):1–15, 1967.

[Rod85] François Rodier. Intégrabilité locale des caractères du groupe $\mathrm{GL}(n, k)$ où k est un corps local de caractéristique positive. *Duke Math. J.*, 52(3):771–792, 1985.

[Ser65] Jean-Pierre Serre. *Lie algebras and Lie groups*, volume 1964 of *Lectures given at Harvard University*. W. A. Benjamin, Inc., New York-Amsterdam, 1965.

[Ser68] Jean-Pierre Serre. *Corps locaux*. Hermann, Paris, 1968. Deuxième édition, Publications de l'Université de Nancago, No. VIII.

[Spr66] Tonny A. Springer. Some arithmetical results on semi-simple Lie algebras. *Inst. Hautes Études Sci. Publ. Math.*, (30):115–141, 1966.

[Spr98] Tonny A. Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1998.

[SS70] Tonny A. Springer and Robert Steinberg. Conjugacy classes. In *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, Lecture Notes in Mathematics, Vol. 131, pages 167–266. Springer, Berlin, 1970.

[SS97] Peter Schneider and Ulrich Stuhler. Representation theory and sheaves on the Bruhat-Tits building. *Inst. Hautes Études Sci. Publ. Math.*, (85):97–191, 1997.

[ST16] Sug Woo Shin and Nicolas Templier. Sato-Tate theorem for families and low-lying zeros of automorphic L-functions. *Invent. Math.*, 203(1):1–177, 2016. Appendix A by Robert Kottwitz, and Appendix B by Raf Cluckers, Julia Gordon and Immanuel Halupczok.

[Ste74] Robert Steinberg. *Conjugacy classes in algebraic groups.* Lecture Notes in Mathematics, Vol. 366. Springer-Verlag, Berlin-New York, 1974. Notes by Vinay V. Deodhar.

[Tit79] Jacques Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, RI, 1979.

[Tsa15] Cheng-Chiang Tsai. Inductive structure of shalika germs and affine springer fibers. *arXiv:1512.00445v1*, 2015.

[Wit15] Julius Witte. Local Summability of Characters on p-adic Reductive Groups. *ArXiv: 1506.07295*, 2015.

[Wit17] Julius Witte. Nilpotent orbits: finiteness, separability and Howe's conjecture. *Journal of Lie Theory*, 27(1):001–042, 2017.

[Yu01] Jiu-Kang Yu. Construction of tame supercuspidal representations. *J. Amer. Math. Soc.*, 14(3):579–622 (electronic), 2001.

Index

Samenvatting

Alvorens het proefschrift wordt samengevat, eerst een korte verhandeling over de vraag "Wat is wiskunde?"

De eerste keer dat men in aanraking komt met het woord "wiskunde" is waarschijnlijk op de middelbare school. Het vak ligt in het verlengde van het basisschoolvak rekenen. In plaats van dat men de tafels en optellen leert, leert men hoe men het minimum van een functie uitrekent, hoe je vergelijkingen oplost, zoals bijvoorbeeld in de volgende redactiesom:
"Willy en Hiske zijn een weekendje in Amsterdam en bezoeken samen met hun dochter Anki een museum. Willy krijgt een 65+ korting van 20 procent en Hiske krijgt vanwege haar Museumjaarkaart een korting van 15 procent. Anki is helaas een 'morekop' en krijgt geen korting. Omdat Hiske al voor de benzine heeft betaald, betaalt Willy ook haar kaartje. Hij moet 16,5 euro afrekenen aan de kassa. Hoeveel moet Anki voor haar kaartje betalen?",
en een beetje kansrekening en meetkunde. Dit is waarschijnlijk de aanleiding voor het vooroordeel dat wiskunde het uitrekenen van sommetjes[1] is.

In de eerste jaren van de studie wiskunde wordt dit vooroordeel hardnekkig de kop ingedrukt. Het gaat voornamelijk om het bewijzen van rekenregels en methodes die bekend zijn van de middelbare school of de studie natuurkunde: van afgeleiden berekenen tot integralen uitrekenen. Behalve bewijzen leert men ook nieuwe methodes om dingen uit te rekenen of, misschien beter gezegd, te bestuderen. Meestal maken deze nieuwe methodes gebruik van nieuwe wiskundige objecten, zoals bijvoorbeeld de negatieve getallen worden geintroduceerd om een zinnig antwoord te kunnen geven om vragen als: "vandaag is het 2 graden kouder dan gisteren toen het 1 graad was, hoeveel graden is het vandaag?"
Wanneer de nieuwe wiskundige objecten goed gekozen zijn, duiken ze ook op in oplossingen van andere vraagstukken. Dan worden de nieuwe wiskundige objecten zelf onderwerp van vraagstukken en worden er weer nieuwe methodes bedacht met als gevolg nieuwere wiskundige objecten. Bijvoorbeeld: bestaat er een getal zodanig dat het kwadraat gelijk is aan -1? Maar ook vragen als: bestaat er een methode om met passer en liniaal een hoek in drie gelijke hoeken te delen? Dit proces herhaalt zich typisch enkele malen. Hierdoor worden de bestudeerde objecten abstracter, dat wil meestal zeggen algemener toepasbaar en tegelijkertijd verder verwijderd van de alledaagse werkelijkheid. Net zoals deze schrijver nu wat afgedwaald is van zijn hoofdpunt.

[1] Het nadeel van het gebrek aan rekenen in de wiskunde is dat het optellen van punten bij verscheidene spelletjes mij nu slechter afgaat dan toen ik 10 jaar was. Ter illustratie, ik heb tijdens het schrijven van deze samenvatting meer gerekend dan het voorgaande half jaar voor mijn onderzoek.

Buitenstaanders denken vaak dat wiskunde rekenen is en voor eerstejaars wiskundigen wordt voornamelijk op de noodzaak van het bewijzen gehamerd. Nu is wiskunde een beetje van beide. Het hoofddoel is het uitrekenen/bestuderen van sommetjes/wiskundige objecten. Aan de ene kant bedenkt men methodes om de sommetjes uit te rekenen en aan de andere kant moet men bewijzen dat deze methodes altijd werken. Soms is er niet een echt duidelijk onderscheid tussen de methode en het bewijs dat de methode werkt: zoals bij het uitrekenen hoeveel Anki voor haar kaartje moet betalen. Het komt echter ook voor dat een methode relatief makkelijk is, maar een bewijs wezenlijk verschilt van de methode. Zo kan men de vraag "Waar ligt het dal van het dal-parabool $f(X) = X^2 - bX + c$?" beantwoorden met "bij $X = \frac{b}{2}$." Een mogelijk bewijs zou kunnen zijn:
Zij $X' = \frac{b}{2} + Z$, dan

$$f(X') = Z^2 + \left(\frac{b}{2}\right)^2 - \frac{b^2}{2} + c = Z^2 + f\left(\frac{b}{2}\right) \geq f\left(\frac{b}{2}\right).$$

Dus voor alle X geldt dat $f(X)$ groter of gelijk is aan $f\left(\frac{b}{2}\right)$. Daarom ligt het dal van het dal-parabool bij $X = \frac{b}{2}$.
Dit bewijs is een voorbeeld van wat weleens vaker voorkomt in de wiskunde: dat het antwoord op de vraag je een idee geeft hoe je zou kunnen bewijzen dat het antwoord klopt. Dit is helaas niet het geval met de vragen die bestudeerd worden in dit proefschrift.

Na deze korte verhandeling over wiskunde in het algemeen, volgt nu een introductie tot p-adische getallen.

Een deel van het wiskundig onderzoek houdt zich bezig met de natuurlijke getallen, de getallen waarmee je dingen telt:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots, 100, \ldots.$$

Deze tak van sport wordt ook wel de getaltheorie genoemd. Een priemgetal is een natuurlijk getal ongelijk aan 1 dat alleen deelbaar is door 1 en zichzelf. De eerste priemgetallen zijn:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47^{1}.$$

Een van de mooie eigenschappen van de natuurlijke getallen is dat elk natuurlijk getal groter dan 1 precies op één manier, op de volgorde na, geschreven kan worden als product van priemgetallen. De priemfactorontbinding[2] van een getal is het schrijven van dat getal als product van priemgetallen. Zo is bijvoorbeeld $2 \cdot 3$ de priemfactorontbinding van 6 en $2 \cdot 2 \cdot 13$ die van 52.
Nu houdt de getaltheorie zich niet alleen bezig met de priemgetallen, maar ook met het

[1]De n-de lezer zonder master in de wiskunde of een vergelijkbare wiskunde-opleiding die het volgende priemgetal naar priem2357@gmail.com stuurt, krijgt van de auteur een beloning ter waarde van $\left\lfloor \frac{1000}{2^n} \right\rfloor$ eurocent. (Deelname is alleen open voor bekenden van de auteur en tot mei 2017. Meerdere inzendingen per deelnemer zijn niet toegestaan; proeflezers zijn uitgesloten van deelname; bij twijfel beslist de auteur.)

[2]Elke deelnemer van vorige prijsvraag kan zijn beloning verdubbelen door gelijktijdig de priemfactorontbinding van het getal 15728640 op te sturen.

oplossen van vergelijkingen. Een klassiek voorbeeld is de volgende vraag: voor welke natuurlijke getallen A, B en C geldt

$$A^2 + B^2 = C^2\,?$$

Om dit soort vergelijkingen op te lossen, kan het soms handig zijn om nieuwe getallen te introduceren. Enkele voorbeelden zijn de gehele, rationale en reële getallen. De gehele getallen zijn de natuurlijke getallen samen met

$$0, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10, \dots, -100, \dots.$$

De rationale getallen zijn de breuken waar de teller en noemer een geheel getal is en de noemer ongelijk aan nul. De reële getallen komen overeen met de getallen op de getallenlijn. Een reëel getal wordt meestal opgeschreven als een getal met oneindig veel decimalen achter de komma.

Elk reëel getal kan men willekeurig precies benaderen door een rationaal getal. Het getal π is bijvoorbeeld bijna gelijk aan het getal

$$3,141592653589793$$

dat gelijk is aan de breuk

$$\frac{3141592653589793}{1000000000000000}.$$

Eigenlijk worden de reële getallen op deze manier geconstrueerd. De reële getallen worden min of meer gedefineerd als de getallen die je kan schrijven als een limiet van een rijtje reële getallen.[1] De decimale schrijfwijze van een reëel getal is als volgt:

$$a_{-2}a_{-1}a_0, a_1a_2a_3 \ldots \leftrightarrow a_{-2} \cdot 10^2 + a_{-1} \cdot 10^1 + a_0 + a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + a_3 \cdot 10^{-3} + \ldots$$

Behalve de reële getallen worden ook p-adische getallen gebruikt voor het bestuderen van vergelijkingen. Hier staat p voor een priemgetal. Laten we als voorbeeld naar het priemgetal 3 kijken. De 3-adische getallen kunnen net als de reële getallen worden opgeschreven als een rijtje getallen met oneindig veel decimalen achter de komma:

$$a_{-2}a_{-1}a_0, a_1a_2a_3 \ldots$$

Hier is elke a_n gelijk aan $0, 1$ of 2. De betekenis is echter verschillend van die van de decimale schrijfwijze van een reel getal. Bovenstaand getal correspondeert met

$$a_{-2} \cdot 3^{-2} + a_{-1} \cdot 3^{-1} + a_0 + a_1 \cdot 3^1 + a_2 \cdot 3^2 + a_3 \cdot 3^3 + \ldots$$

Bij de 3-adische getallen is er een som van machten van 3 i.p.v 10. Het grootste verschil is echter dat de machten bij de p-adische getallen oplopen i.p.v. aflopen. Elk geheel getal kunnen we ook p-adisch opschrijven. Laten we 50 opschrijven als 3-adisch getal. Dit doen

[1] Voor de geínteresseerde lezer: het zijn de Cauchyrijtjes. Een rij rationale getallen $a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$ is een Cauchyrij als voor alle $\epsilon > 0$ er een $N > 0$ bestaat zodanig dat voor alle $n, m > N$, $|a_n - a_m| < \epsilon$. Informeel gezegd: de afstand tussen de rationale getallen ver in de rij wordt steeds kleiner.

we door achtereenvolgens te delen door 3 met rest: $50 = 2 + 16 \cdot 3$. Nu gaan we 16 delen door 3 met rest: $16 = 1 + 5 \cdot 3$. En als laatste $5 = 2 + 3$. Dus

$$50 = 2 + 3 + 2 \cdot 3^2 + 3^3 = 2,121.$$

Het optellen en vermenigvuldigen van 3-adische getallen gaat in een andere volgorde dan bij de reële getallen: van links naar rechts. Laten we simpel beginnen: wat is 1+2. We weten dat dit gelijk is aan 3. In de 3-adische schrijfwijze is dat gelijk aan $3 = 1 \cdot 3 = 0,1$. Dus we schrijven $1 + 2 = 0,1$. Wat is nu $2 + 2$?

$$2 + 2 = 4 = 1 + 3 = 1,1$$

Laten we 50 bij $11(= 2,01)$ optellen in de 3-adische schrijfwijze:

$$\begin{array}{rcccc} & & 1 & & 1 \\ & 2, & 1 & 2 & 1 \\ + & 2, & 0 & 1 & \\ \hline & 1, & 2 & 0 & 2 \end{array}$$

Vermenigvuldigen gaat op dezelfde manier. Laten we $5 \cdot 10$ uitrekenen:

$$\begin{array}{rcccc} & 2, & 1 & & \\ \times & 1, & 0 & 1 & \\ \hline & 2, & 1 & & \\ + & 0, & 0 & 2 & 1 \\ \hline & 2, & 1 & 2 & 1 \end{array}$$

Net als in de reële getallen zijn twee 3-adische getallen dichtbij elkaar als het begin van de schrijfwijze overeenkomt: 2,01201 ligt dicht bij 2,01 en nog dichter bij 2,012. Dit betekent dus dat 3^{20} dicht bij 0 ligt in de 3-adische getallen. Ook kan men de rationale getallen terugvinden in de 3-adische getallen. De 3-adische getallen kunnen we net als de reële getallen beschouwen als de limieten van rijtjes rationale getallen. Elk 3-adisch getal kan benaderd worden door een rationaal getal. Dit gaat op dezelfde manier als bij de reële getallen:

$$\frac{\sqrt{7}}{9} \approx 211,2022011020001201.$$

In dit geval correspondeert 211,2022011020001201 met het rationale getal $\frac{488245388}{9}$.
De valuatie van een 3-adisch getal is “het aantal nullen achter de komma”. Dat wil zeggen de valuatie van 1 is 0, de valuatie van 0,2 is 1 en de valuatie van 0,001 is 3. De valuatie van een 3-adisch getal a wordt genoteerd met $\nu(a)$. Dus $\nu(0,21201\ldots) = 1$. Door de regels voor vermenigvuldigen van 3-adische getallen is er een mooie relatie tussen de valuatie en de vermenigvuldiging:
De valuatie van een product van twee 3-adische getallen is gelijk aan de som van de valuaties van de twee 3-adische getallen.
Uitgedrukt in formules komt dit neer op $\nu(ab) = \nu(a) + \nu(b)$. De 3-adische getallen worden onder andere gebruikt voor bewijzen van stellingen over natuurlijke getallen.

Voorbeeld. Er bestaan geen natuurlijke getallen a, b, zodat $a^2 = 3b^2$.
Stel a, b zijn natuurlijke getallen zodanig dat $a^2 = 21b^2$. Laten we a, b en 21 als 3-adische getallen bekijken. De valuatie van a^2 is twee keer de valuatie van a. Dus $\nu(a^2)$ is even. De valuatie van b^2 is om dezelfde reden ook even. Het getal $21 = 3 + 2 \cdot 3^2$, dus 3-adisch schrijven we $0,12$. Omdat de valuatie van $0,12$ is gelijk aan 1, is de valuatie van $\nu(21b^2)$ oneven. Dus $\nu(a^2) = \nu(21b^2)$ is tegelijkertijd even als oneven. Dit kan natuurlijk niet. Daarom bestaan er geen natuurlijke getallen a, b, zodat $a^2 = 21b^2$.

Het kan dus handig zijn om vergelijkingen over de natuurlijke of rationale getallen te bekijken over de 3-adische getallen.

De p-adische getallen voor de overige priemgetallen werken net zo als bij de 3-adische getallen. De cijfers die gebruikt worden om een p-adisch getal te representeren zijn dan $0, \ldots, p-1$. De p-adische getallen zijn een combinatie van de analytische kant van de reële getallen met de getaltheoretische kant van de priemgetallen.

Voor elk priemgetal is er ook een ander getallenstelsel dat in dit proefschrift een rol speelt. Dit zijn de formele Laurentreeksen. Laten we nu naar het priemgetal 5 kijken. We noteren de formele Laurentreeksen voor het priemgetal 5 met $\mathbb{F}_5((X))$. De "getallen" kunnen op dezelfde manier genoteerd worden als voor de 3-adische getallen:

$$a_{-3}a_{-2}a_{-1}a_0, a_1a_2a_3\ldots,$$

met elke a_i gelijk aan $0, 1, 2, 3$ of 4. In dit geval correspondeert de schrijfwijze met het 'getal' (d.w.z. Laurentreeks)

$$a_{-3}X^{-3} + a_{-2}X^{-2} + a_{-1}X^{-1} + a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots.$$

Het optellen en vermenigvuldigen gaat bijna op dezelfde wijze als bij de 3-adische getallen. Het belangrijkste verschil is dat we niet doen aan het "onthouden bij het optellen en vermenigvuldigen". We kijken alleen maar naar de rest bij delen door 5. Dus $3+4$ wordt 2, want 7 heeft rest 2. Evenzo is $0,2 + 0,3 = 0$, omdat $2 + 3 = 0 + 5$. Laten we $4,32$ optellen bij $1,14$:

$$\begin{array}{rrrr} & 4, & 3 & 2 \\ + & 1, & 1 & 4 \\ \hline & 0, & 4 & 1 \end{array}$$

Ter contrast de optelling van $4,32$ bij $1,14$ als 5-adische getallen:

$$\begin{array}{rrrrr} & & 1 & 1 & 1 \\ & 4, & 3 & 2 & \\ + & 1, & 1 & 4 & \\ \hline & 0, & 0 & 2 & 1 \end{array}$$

Bij de 5-adische getallen wordt als de som over de 5 gaat er 1 opgeteld bij de som van de cijfers eentje verder naar rechts; dit "onthouden" laten we achterwegen bij de formele Laurentreeksen. Ook het vermenigvuldigen gaat zo: $3 \cdot 4 = 12 = 2 + 2 \cdot 5$, dus $3 \cdot 4 = 2$.

Het vermenigvuldigen van $1,234$ met $4,213$ kan als volgt uitgerekend worden:

$$\begin{array}{cccccccc}
 & 1, & 2 & 3 & 4 & & & \\
\times & 4, & 2 & 1 & 3 & & & \\
\hline
 & 4, & 3 & 2 & 1 & & & \\
 & 0, & 2 & 4 & 1 & 3 & & \\
 & 0, & 0 & 1 & 2 & 3 & 4 & \\
+ & 0, & 0 & 0 & 3 & 1 & 4 & 2 \\
\hline
 & 4, & 0 & 2 & 2 & 2 & 3 & 2
\end{array}$$

De p-adische getallen en de formele Laurentreeksen zijn voorbeelden van niet-Archimedische lokale lichamen.

Na al deze formules kan nu ook de overgebleven 5 procent van de leken na deze alinea afhaken met lezen. De inhoudelijke samenvatting van het proefschrift is namelijk zo doordrenkt met wiskundig jargon dat deze alleen door experts in reductieve p-adische groepen te verteren is.

In dit proefschrift worden enkele meetkundige aspecten van reductieve p-adische groepen besproken met een oogmerk op toepassingen in de representatietheorie van deze groepen.

Door het bestuderen van de vaste punten van een compact regulier semisimpel element in het gereduceerde gebouw krijgen we een afschatting van de absolute waarde van het karakter van een representatie waarvan de lengte eindig is. Op vergelijkbare wijze worden baan-integralen van reguliere semi-simpele elementen afgeschat. Deze afschattingen zijn klein genoeg om aan te tonen dat het karakter lokaal integreerbaar is op elke conjugatieklasse van een maximale torus die een maximaal gespleten torus bevat.

Tevens bestuderen we de meetkunde van de nilpotente banen in de Lie-algebra. We gaan voor een gespleten reductieve groep na wanneer alle banen separabel zijn en wanneer er maar eindig veel nilpotente banen zijn. Ook stellen we vast voor welke gespleten reductieve groepen Howe's vermoeden waar is.

Voor sommige rijtjes van supercuspidale representaties van de algemene lineaire groep tonen we aan dat, gegeven een regulier semi-simpel element, de absolute waarde van het karakter in dat element gedeeld door de graad van de representatie naar nul convergeert naarmate de graad van de representatie groter wordt.

In hoofdstuk 6 wordt aangetoond dat gegeven een reductieve p-adische groep G er een constante C bestaat, die alleen afhangt van het worteldatum van de groep, zodanig dat voor elk compact element $g \in G$ en elk punt p in het gereduceerde gebouw van G er een door g vastgelaten punt p_0 in het gebouw bestaat, zodat $d(p, p_0) \leq Cd(p, gp)$.